BBA 1304

Business Mathematics

Study Module

স্কুল অব বিজনেস

SCHOOL OF BUSINESS

বাংলাদেশ উন্মুক্ত বিশ্ববিদ্যালয়

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Business Mathematics

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Lesson – 1: Meaning, Methods, and Types of Set

After studying this lesson, you should be able to:

- Identify sets and its elements;
- Apply the methods of describing a sets;
- Define and explain different types of sets.

Introduction

Mathematics speaks in the language of sets because it lies at the foundations of mathematics. Set is an undefined term, just as point and line are undefined in geometry.

Meaning of Sets and Element

A set is understood to be a collection of objects. In other way, a set is a collection of definite and well distinguished objects. Each object belonging to a set is known as an element of the set.
Generally capital letters *A, B, C, X, Y*....etc. are used to denote a set and small letters *a, b, c, x, y,* ...etc are used to denote elements of a set.

A set may be described by listing its members/elements between the symbols {and}, which are called set braces. Thus, the expression {1, 2, 3, 4} is read as: The set of 1, 2, 3 and 4. The elements of the set are 1, 2, 3 and 4. The symbol for set elements is ∈. Thus 1∈ {1, 2, 3, 4} is read as: 1 is an element of {1, 2, 3, 4}. The symbol ∉ is the negation of ∈. Thus 6 ∉ {1, 2, 3, 4} is read as: 6 is not an element of {1, 2, 3, 4}.

Methods of Describing a Set

A set can be described in the following two ways:

(1) Tabular Method: In this method, all the elements of the set are enclosed by set braces. For example,

(a) A set of vowels; A = {a, e, i, o, u}

(b) A set of even numbers; A = {2, 4, 6, ...}

(c) A set of first five letters of alphabet; A = {a, b, c, d, e}

(d) A set of odd numbers between 10 and 20; A = {11, 13, 15, 17, 19}

(2) Selector / Set-builder Notation Method: In this method, elements of the set can be described on the basis of specific characteristics of the elements. For example, let if x is the element of a set, then the above four sets can be expressed in the following way:

(a) A = {*x* | *x* is a vowel of English alphabet}

(b) A = {*x* | *x* is an even number}

(c) A = {*x* | *x* is a letter of the first five alphabet in English}

(d) A = {*x* | *x* is an odd number between 10 and 20}

In this case, the vertical line "|" after *x* is to be read as "such that".

Types of Sets

A set can be classified on the basis of special features of elements. There are different types of sets which are discussed below:

(i) Null, Empty or Void Set: A set having no element is known as null, empty or void set. It is denoted by Ø. For example,

(i) A = {*x* | *x* is an odd integers divisible by 2}

(ii) A = $\{x \mid x^2 = 4, x \text{ is odd}\}$

A is the empty set in the above two cases.

(ii) Finite Set: A set is finite if it consists of a specific number of different elements, i.e. the counting process of the different members/elements of the set can come to an end. For examples,

(i) A = {1, 2, 3, 4, 5} (ii) A = {a, e, i, o, u}

then the sets are finite, because the elements can be counted by a finite number.

(iii) Infinite Set: If the elements of a set cannot be counted in a finite number, the set is called an infinite set. For example,

(a) Let A = {1, 2, 3, 4 ......}

(b) Let A = {x | x is a positive integer divisible by 5},

then the sets are infinite, as the process of counting the elements of these sets would be endless.

(iv) Sub Sets: If every element in a set A is also the element of a set B, then A is called a subset of B. We denote the relationship by writing A $\subseteq$ B, which can also be read as "A is contained in B." For example,

A = {1, 2, 3, 4......}

B = {x | x is a positive even number}

C = {x | x is a positive odd number}

In this case B $\subseteq$ A and C $\subseteq$ A, because all the positive even and odd numbers are included in the set A.

(v) Proper Subset: Since every set A is a subset of itself, we call B is a proper subset of A if B is a subset of A and B is not equal to A. If B is a proper subset of A, it can be represented symbolically as B $\subset$ A. For example,

A = {a, b, c, d}, B = {a, c, b, d, c, a}

C = {a, c, d, a, d, a}

In this case, C $\subset$ A and C $\subset$ B, because the elements of C set are included in the sets A and B, but the element 'b' in of A and B sets is not element of C set.

(vi) Equal Sets: Two sets A and B are said to be equal if every element which belongs to A also belongs to B, and if every element which belongs to B, also belongs to A. We denote the equality of sets A and B by 'A = B'. For example, let A = {2, 3, 4}, B = {4, 2, 3}, C = {2, 2. 3, 4}, then A = B = C, since each element which belongs to any one of the sets also belongs to the other two sets.

(vii) Equivalent Sets: If the elements of one set can be put into one to one correspondence with the elements of another set, then the two sets are called equivalent sets. For example,

Let A = {a, b, c, d, e, f} and B = {1, 2, 3, 4, 5, 6}

In this case, the elements of set A can be put into one to one correspondence with those of set B. Hence the two sets are equivalent. It is denoted by A ≡ B.

(viii) Unit Set/singleton: A set containing only one element is called a unit set or singleton. For example,

(a) A = {a}

(b) B = {x | x is a number between 27 and 34 divisible by 10}

In B set, 30 is the only number between 27 and 34 which is divisible by 10.

(ix) Power Set: The set of all the subsets of a given set A is called the power set of A. We denote the power set of A by P(A). The power set is denoted by the fact that 'if A has n elements then its power set P(A) contains exactly 2^n elements'.

For example, let A = {a, b, c} then its subset are {a}, {b}, {c}, {a,b}, {b,c}, {c,a} {a,b,c}, {Ø}

∴ P (A) = [{a}, {b}, {c}, {a,b), {b,c}, {c,a}, {a,b,c}, {Ø}]

(x) Disjoint Sets: If the sets A and B have no element in common, i.e., if no element of A is in B and no element of B is in A, then we say that A and B are disjoint.

For example, let A = {3, 4, 5} and B= {8, 9, 10, 11}, then A and B sets are disjoint because there is no element common in these two sets.

(xi) Universal Sets: Usually, only certain objects are under discussion at one time. The universal set is the set of all objects under discussion. It is denoted by U or I. For example, in human population studies, the universal set consists of all the people in the world.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. How would you define set? Identify some of the characteristics of sets. Is there any distinction between set and element?
2. Define the following with examples:
   Null set, finite set, infinite set, disjoint set, equal sets, equivalent sets, venn diagram, universal set.
3. (a) What is a subset and proper subset.
   (b) Find the power set of A = {1, 2, 3, 4}
4. List the elements of the following sets:
   (a) The set of all integers whose squares are less than 30;
   (b) The set of integers satisfying the equation $x^2 - 7x + 10 = 0$;
   (c) The set of all positive integers which are divisible by 5 and smaller than 78.
5. State whether each of the following sets is finite or infinite. When the set is finite indicate the number of elements it possesses;
   (a) The set of odd positive integers;
   (b) The set of all integers, whose squares are less than 45,
   (c) The set of integers satisfying the equation $x^2 - 5x + 6 = 0$
   (d) The set of students in your class who are taller than 7 feet.

Lesson – 2: Venn Diagrams

After studying this lesson, you should be able to:

- Draw a Venn diagram of any set;
- Explain the nature of Venn diagram;
- Apply laws of sets for set operations;
- Explain the relationship between sets by using Venn diagram.

Venn Diagrams

Generally Venn diagram is used to help visualize any set and the relationship between sets. It is usually bounded by a circle. With the help of Venn diagram we can easily illustrate various set operations.

Following is the Venn diagram (Fig.1) of three sets A, B and C:

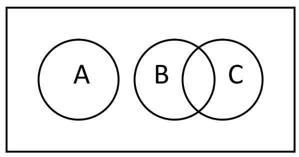


Fig.1

Laws of Algebra of Sets

Basic set operations viz. union, intersection and complement satisfy some laws, known asLaws of Algebra of Sets. We state below these laws of algebra of sets:

1. **Idempotent Laws:** For any set A, we have (i) $A \cup A = A$, (ii) $A \cap A = A$.
2. **Commutative Laws:** For any two sets A and B, we have (i) $A \cup B = B \cup A$, (ii) $A \cap B = B \cap A$.
3. **Associative Laws:** For any three sets A, B and C, we have, (i) $A \cup (B \cup C) = (A \cup B) \cup C$, (ii) $A \cap (B \cap C) = (A \cap B) \cap C$.
4. **Distributive Laws:** (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
5. **De Morgan's Laws:** For any two sets A and B, we have (i) $(A \cup B)^c = A^c \cap B^c$, (ii) $(A \cap B)^c = (A^c \cup B^c)$
6. **Identity Laws:** Let U be the universal set, ϕ be the null set and A be any subset of U. Then, (i) $A \cup U = U$, (ii) $A \cap U = A$,(iii) $A \cup \phi = A$, (iv) $A \cap \phi = \phi$.
7. **Complement Law :** with the same notation given in (6) above, we have (i) $A \cup A^c = U$, (ii) $(A \cup U)^c = \phi$, (iii) $(A^c)^c = A$, (iv) $U^c = \phi$, (v) $\phi^c = U$, where A^c is the complement of A.

Note: We observe similarity is some laws of the set theory with the ordinary algebraic laws of real numbers. If *a, b, c* are real numbers, we have following laws of algebra of numbers:

(i) $a + b = b + a$,
(ii) $a \times b = b \times a$,
(iii) $a + (b + c) = (a + b) + c$
(iv) $a \times (b \times c) = (a \times b) \times c$
(v) $a \times (b + c) = a \times b + a \times c$.

If addition (+) and multiplication (×) notations of algebra of real numbers are replaced respectively by union (∪) and intersection (∩) notations of the set theory and the real numbers *a,* *b, c* are also replaced by the sets A, B, and C respectively, we obtain the following laws of algebra of sets:

(i) $A \cup B = B \cup A$
(ii) $A \cap B = B \cap A$
(iii) $A \cup (B \cup C) = (A \cup B) \cup C$
(iv) $A \cap (B \cap C) = (A \cap B) \cap C$
(v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

But some laws of algebra of sets differ from algebra of real numbers. For example, in ordinary algebra of real numbers, we have, (i) $a + a = 2a$, (ii) $a \times a = a^2$. But in algebra of sets, we have, (i) $A \cup A = A$, (ii) $A \cap A = A$.

In algebra of numbers, addition does not distribute across multiplication, i.e., for three real numbers *a, b* and *c*, $[a + (b \times c)] \neq (a + b) \times (a + c)$.

But in algebra of sets, union distributes across intersection, i.e., for three sets A, B and C we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Example-1:

Using Venn diagram, verify that $A \cap (B \cap C) = (A \cap B) \cap C$

Solution:

LHS: Assume that the rectangular regions in Figs.-2, 3, 4 and 5 represent the universal set U and its subsets A, B and C in each diagram are represented by circular regions.

In Fig.-2, the set A has been shaded by horizontal straight lines and the set $(B \cap C)$ has been shaded by vertical straight lines (i.e., the region common to both the sets B and C). Then by definition, the cross hatched region (i.e., the region where the horizontal and vertical lines intersect) represents the set $A \cap (B \cap C)$. The region representing this set has been shaded separately by slanting lines in Fig.-3.

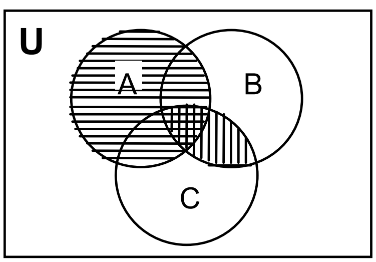


Fig. 2

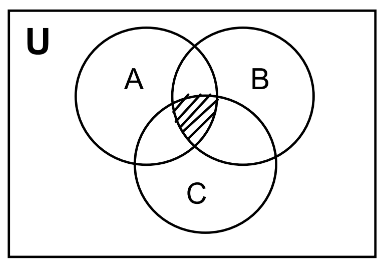


Fig. 3

RHS: In Fig.-4, the set $(A \cap B)$ has been shaded by horizontal lines (i.e., the region common to both the sets A and B) and the set C has been shaded by vertical straight lines. Then by definition, the cross hatched region (i.e., the region where the horizontal and vertical lines intersect) represents the set $(A \cap B) \cap C$. The region representing this set has been shaded separately by slanting lines in Fig.-5.

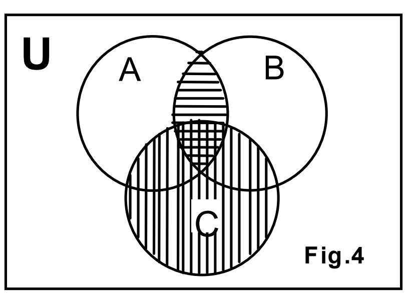


Fig.4

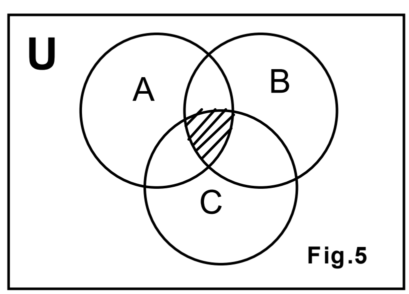


Fig.5

From Figs.-3 and 5, we see that the regions representing the sets [A ∩ (B ∩ C)] and [(A ∩ B) ∩ C] are identical. This verifies that A ∩ (B ∩ C) = (A ∩ B) ∩ C.

Example-2:

Using Venn diagram, verify that A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

Solution:

LHS: Assume that the rectangular regions in Figs.-6, 7, 8 and 9 represent the universal set U and its subsets A, B and C in each diagram are represented by circular regions.

In Fig.-6, the set A has been shaded by cross of horizontal and vertical lines. Set C has been shaded by horizontal straight lines and the set B has been shaded by vertical straight lines (i.e., the region common to both the sets B and C becomes a cross hatched region). Then by definition, the total cross hatched region represents the set A ∪ (B ∩ C).

The region representing this set has been shaded separately by slanting straight lines in Fig.-7.

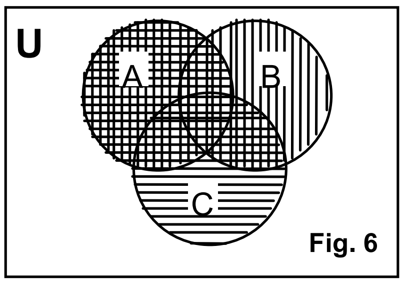


Fig. 6

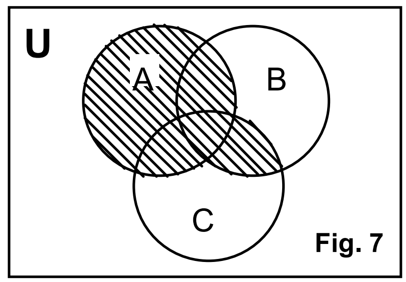


Fig. 7

RHS: In Fig.-8, the set (A ∪ B) has been shaded by vertical straight lines (i.e., the total region enclosed by the sets A and B) and the set (A ∪ C) has been shaded by horizontal straight lines (i.e., the total region enclosed by the sets A and C). Then by definition, the cross hatched region (i.e., the region where the horizontal and vertical lines intersect) represents the set (A ∪ B) ∩ (A ∪ C).The region representing this set has been shaded separately by slanting lines in Fig.-7.

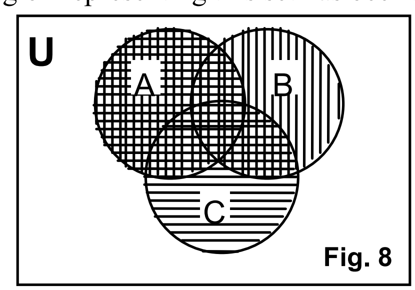


Fig. 8

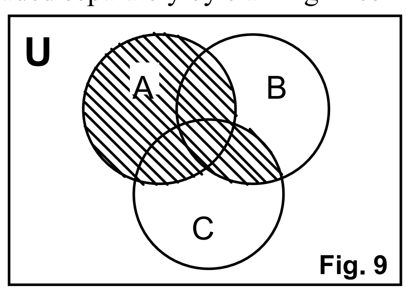


Fig. 9

From Figs.-7 and 9, we see that the regions representing the sets A ∪ (B ∩ C) and (A ∪ B) ∩ (A ∪ C) are identical. This verifies that A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C).

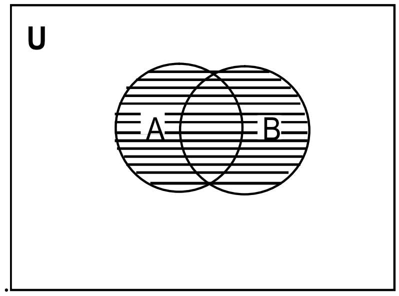
Example-3: [De Morgan's Laws]

Using Venn diagrams, verify that $(A \cap B)^c = A^c \cap B^c$

Solution:

LHS: Assume that the rectangular regions in Figs.-10, 11, 12 and 13 represent the universal set U and its subsets A and B in each diagram are represented by circular regions.

In Fig.-10, the set A ∪ B has been shaded by horizontal straight lines (i.e., the total region enclosed by the sets A and B). Then by definition, the region of the rectangle outside the shaded region represents the set $(A \cap B)^c$(i.e. the complement of A ∪ B). The region represented by $(A \cap B)^c$has been shaded separately by slanting lines in Fig.-11.



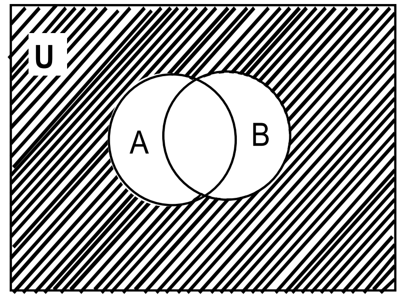
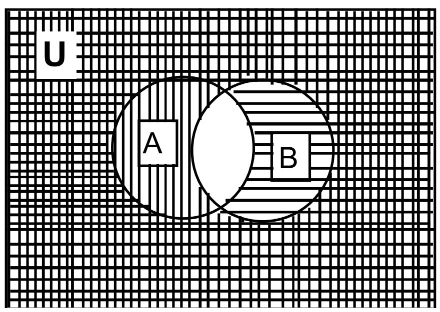


Fig.10 **Fig.11**

RHS: In Fig.-12, the set A^c has been shaded by horizontal straight lines (i.e., theregion of the rectangle outside the set A) and the set B^c has been shaded by vertical straight lines (i.e., the region of the rectangle outside the set B). Then by definition, the cross hatched region (i.e., the region where the horizontal and vertical lines intersect) represents the set $A^c \cap B^c$. The region represented by the set $A^c \cap B^c$has been shaded separately by slanting lines in Fig.-13.



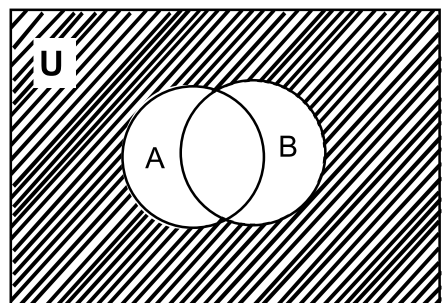


Fig.12 **Fig.13**

From Figs.-11 and 13, we see that the region representing the sets $(A \cup B)^c$and $A^c \cap B^c$ are identical. This verifies that $(A\cup B)^c = A^c \cap B^c$

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. What are the laws of algebra in set theory?
2. Using Venn diagram verify that $A \cup (B \cup C) = (A \cup B) \cup C$
3. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ using Venn diagram.
4. Prove that $(A \cap B)^C = A^C \cup B^C$ using Venn diagram.
5. Using Venn diagram show that $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

Lesson-3:Addition, Subtraction, and Complement of Sets

After studying this lesson, you should be able to:

- Apply the addition operation of sets;
- Apply the subtraction operation of sets;
- Apply the complement operation of sets.

Introduction

Basic set operations will help the mathematician in identifying common elements or uncommon elements or differences of elements between two or more sets. It has been discussed as under:

Union of Sets

The union of sets X and Y is the set of all elements, which belong to X or to Y or to both. We denote the union of X and Y by ($X \cup Y$), which is read as 'X union Y'. The union of X and Y may also be defined concisely by, $X \cup Y = \{x : x \in X \text{ or } Y\}$

Example-1:

Let X = {1, 2, 3, 4, 5, 6} and Y = { 3, 4, 5, 6, 7, 8}

then $X \cup Y$ = {1, 2, 3, 4, 5, 6,7, 8} (According to the tabular method)

$X \cup Y = \{a : a \in I \text{ and } 1 \leq a \leq 8\}$ (According to selector method).

Properties

The important properties of the union of two or more sets are:

- The individual sets composing a union are elements/ members of the union, In other words $X \subseteq (X \cup Y)$ and $Y \subseteq (X \cup Y)$
- It has an identity property in an empty/null set. $\therefore$ $X \cup Ø = X$, for every set X.
- Union of a set with itself is the set itself, i.e., $X \cup Y = X$, for every set X.
- It has a commutative property, i.e., for any two sets X and Y, $X \cup Y = Y \cup X$
- It has an associative property, i.e., for any three sets X, Y and Z, $(X \cup Y) \cup Z = X \cup (Y \cup Z)$
- If $Y \subseteq X$, then $X \cup Y = X$ and if $x \subseteq y$, then $X \cup Y = Y$.
- $X \cup Y = Ø$, then $X = Ø$ and $Y = Ø$, in other words, both are null sets.
- $X \cap Y$ is the proper subset of X and X is the proper subset of $X \cup Y$. i.e., $(X \cap Y) \subset X \subset (X \cup Y)$.

Intersection of Sets:

The intersection of sets X and Y is the set of elements, which are common to X and Y, that is, those elements which belong to X and which also belong to Y. We denote the intersection of X and Y by $X \cap Y$, which is read as 'X intersection Y'.

The intersection of X and Y may also be defined concisely by $X \cap Y = \{b : b \in X, b \in y\}$

Example-2:

Let X = {2, 3, 4, 5, 6, 7) and Y = {3, 4, 5, 6, 7, 8, 9}

Then $X \cap Y$ = {3, 4, 5, 6, 7} (According to tabular method)

$X \cap Y = \{b : b \in I \text{ and } 3 \leq b \leq 7\}$ (According to selector method)

Properties

The important characteristics of intersection of sets are as follows:

- $X \cap Y$ is the subset of both the set X and the set Y, i.e. $(X \cap Y) \subseteq X$ and $(X \cap Y) \subseteq Y$.
- Intersection of any set with an empty set is the null set, i.e., $X \cap Ø = Ø$ for every set X.
- Intersection of a set with itself is the set itself, i.e. $(X \cap Y) = X$, for every set X.
- Intersection has commutative property, i.e., $X \cap Y = Y \cap X$.
- Intersection has associative property. For any three sets X, Y and Z, $(X \cap Y) \cap Z = X \cap (Y \cap Z)$

- If $X \subseteq Y$, then $X \cap Y = X$ and $Y \subseteq X$, than $X \cap Y = Y$. For example, if X = {2, 3} and Y = {2, 3, 4, 5, 6}, then X is the subset of Y, i.e., $X \subseteq Y$. In this case $X \cap Y$ = {2, 3}, because 2 and 3 are the common elements of X and Y sets. Therefore, $X \cap Y = X$.
- If $X \subseteq Y$ and $Y \subseteq X$ then $X \subseteq (Y \cap Z)$; because $Y \subseteq Z$ then $Y \cap Z = Y$:

Distributive Laws of Unions and Intersections of Sets

The distributive laws of unions and intersections of the sets can be illustrated as under:

(i) The laws of the algebra of sets mentioned that the union distributes over intersection which is not possible in ordinary algebra,
 i.e., $X \cup (Y \cup Z) = (X \cup Y) \cap (X \cup Z)$
 Let X = {a, b, c, d, e}; Y = {c, d, e, f, g} and Z = {e, f, g, h, i}
 then $(Y \cap Z)$ = {e, f, g} and $X \cup (Y \cap Z)$ = {a, b, c, d, e, f, g}
 On the other side, $X \cup Y$ = {a, b, c, d, e, f, g}; $X \cup Z$ = {a, b, c, d, e, f, g, h, i}
 then $(X \cup Y) \cap (X \cup Z)$ ={a, b, c, d, e, f, g}
 So, $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

(ii) The algebra of sets can be expressed that the intersection distribute over the union which is also there in ordinary algebra. i.e., $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.
 Let X = {1. 2, 3, 4}, Y = {2, 3, 4, 5} and Z = {3, 4, 5, 6}
 Then $(Y \cap Z)$ = {2, 3, 4, 5, 6} and
 $X \cap (Y \cup Z)$ = {2, 3, 4}
 On the other hand,
 $(X \cup Y)$ = {2, 3, 4}; $(X \cap Z)$ = {3, 4}
 $\therefore (X \cap Y) \cup (X \cap Z)$ = {2, 3, 4}
 So, $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

Complement of a Set

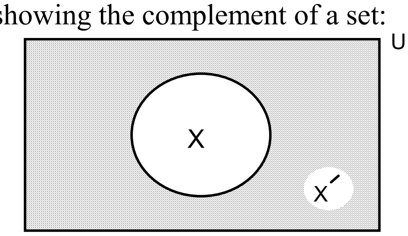
The complement of a set X is the set of elements which do not belong to X, that is, the difference of the universal set U and X. We denote the complement of X by X^C or X′. The complement of X may also be defined concisely by, $X^C = U - X = \{x : x \in U,\ x \notin X\}$.

Example-3:

Let U = {1, 2, 3, 4, 5, 6, 7} and X = {2, 3, 4, 5}

Then $X^C = U - X$ = {1, 6, 7}

The following Venn diagram showing the complement of a set:



The shaded region is X^C or $X' = (U - X)$

Properties

The important properties of complement of a set are:

- The intersection of a set X and its complement X′ is a null set, i.e., $X \cap X' = \emptyset$.
- The union of a set X and its complement X′ is the universal set, i.e., $X \cup X' = U$.
- The complement of the universal set is the empty set and the complement of the empty set is the universal set. Symbolically, $U' = \emptyset$ and $\emptyset' = U$.
- The complement of the complement of a set is the set itself. Symbolically, $(X')' = X$.
- If X is the proper subset of Y, then the complement of Y set is the proper subset of complement of X set. Symbolically, if $X \subset Y$, then $Y' \subset X'$.
- Expansion or contraction of sets is possible by taking into account the complements of a set. Forexample, $(X \cap Y) \cup (X \cap Y') = X$, and $(X \cup Y) \cap (X \cup Y') = X$.

Example-4:

Let A = {1, 2, 3, 4}, B = {2, 4, 6, 8} and C = {3, 4, 5, 6}. Find (i) $A\cup B$, (ii) $A\cup C$, (iii) $B\cup C$, (iv) $B\cup B$, (v) $(A\cup B)\cup C$, (vi) $A\cup (B\cup C)$.

Solution:

(i) $A \cup B$ = {1, 2, 3, 4, 6, 8}
(ii) $A \cup C$ = {1, 2, 3, 4, 5, 6}
(iii) $B \cup C$ = {2, 3, 4, 5, 6, 8}
(iv) $B \cup B$ = {2, 4, 6, 8}
(v) $(A\cup B)\cup C$ = {1, 2, 3, 4, 5, 6, 8}
(vi) $A\cup (B\cup C)$ = {1, 2, 3, 4, 5, 6, 8}

Example-5:

Let A = {2, 3, 4, 5}, B = {3, 5, 7, 8} and C = {4, 5, 6, 7, 8}

Find (i) $A\cap B$, (ii) $A\cap C$, (iii) $B\cap C$, (iv) $B\cap B$, (v) $(A\cap B)\cap C$, (vi) $A\cap (B\cap C)$.

Solution:

(i) $A \cap B$ = {3, 5}
(ii) $A \cap C$ = {4, 5}
(iii) $B \cap C$ = {5, 7, 8}.
(iv) $B \cap B$ = {3, 5, 7, 8}
(v) $(A\cap B)\cap C$ = {5}
(vi) $A\cap (B\cap C)$ = {5}.

Example-6:

Let A = {1, 2, 3, 4}, B = {2, 4, 6, 8} and C = {3, 4, 5, 6}. Find (i) A – B; (ii) C – A; (iii) B – C; (iv) B – A; (v) B – B.

Solution:

(i) A – B = {1, 3}
(ii) C – A = {5, 6}
(iii) B – C = {2, 8}
(iv) B – A = {6, 8}
(v) B – B = {ϕ}.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Define the following with examples:
   (a) Union of sets, (b) Intersection of sets, and (c) Complement of a set.
2. Let the universal set and sets A, B and C are as follows:
   U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
   A = {1, 2, 3, 5}
   B = {2, 5, 6, 8}
   C = {5, 6, 8, 9, 10}
   Find (i) $A\cup B\cup C$; (ii) $(A\cup B\cup C)'$; (iii) $A\cap B\cap C$; (iv) C'; (v) B'; (vi) A'.
3. Let U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
   A = {1, 2, 5, 7}, B = {0, 1, 4, 6}, C = {3, 4, 5, 7}
   Show that $(A\cup B\cup C)' = A'\cap B'\cap C'$
4. If the universal set U = $\{x : x\in N \text{ and } 1 \le x \le 10\}$
   A = $\{x : x\in N \text{ and } 1 \le x \le 8\}$
   B = {x : x is a natural number, which is less than 10 and divisible by 3}
   C = {1, 2, 3, 5, 6}
   Find (i) A'; (ii) $A\cup B$; (iii) $A\cap C$; (iv) $(A\cup C)'$; (v) $B'\cap C$.

Lesson-4: Difference and Product of Sets

After studying this lesson, you should be able to:

- State the difference of sets;
- State the product of sets;
- Explain the presentation of sets with corresponding set notation.

Difference of Two Sets

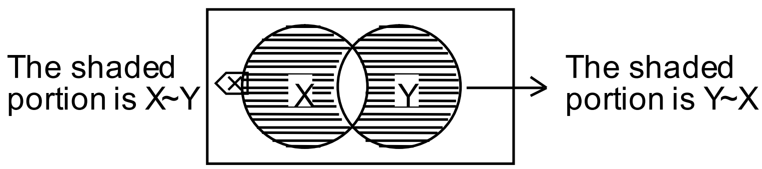
The difference of set Y from set X is the set of elements, which belong to X but which do not belong to Y. We denote the difference of X and Y by (X ~ Y), which is read as: X difference Y, or simply, 'X minus Y'. The difference of X and Y may also be defined concisely by, $X \sim Y = \{a : a \in X, a \notin Y\}$.

For example**:** Let X = {a, b, c, d, e, f} and Y = {d, e, f, g, h}

Then X ~ Y = {a, b, c}

and Y ~ X = {g, h}

The difference of two set can be shown by Venn diagram as under:



Properties

The important properties of the difference of two sets are as under:

- $X - Y$ is the subset of X, i.e., $(X - Y) \subseteq X$ and $(Y - X)$ is the subset of Y, i.e., $(Y - X) \subseteq Y$.
- $(X - Y)$, $(X \cap Y)$ and $(Y - X)$ are mutually disjoints.
- $X - (X - Y) = (X \cap Y)$ and $Y - (Y - X) = X \cap Y$.

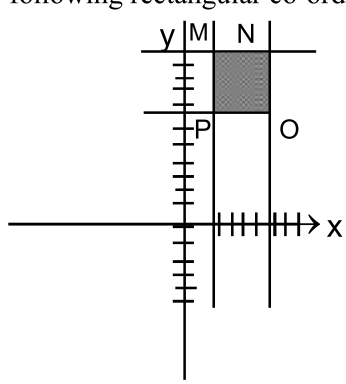
Product of Two Sets

Let X and Y be two sets. The product of sets X and Y consists of all ordered pairs where $\{(x, y): x \in X \text{ and } y \in Y\}$. It is denoted by $(X \times Y)$, which is read as "X cross Y". The product of sets X and Y may also be defined concisely by, $(X \times Y) = \{(x, y) : x \in X, y \in Y\}$.

The product of sets $(X \times Y)$ is also called Cartesian product of X and Y.

For example: Let X = {1, 2, 3} and Y = {5, 6, 7}

Then X.Y = {(1,5), (1,6), (1,7), (2,5) (2,6) (2,7), (3,5), (3,6) (3,7)}. The Cartesian product of X and Y sets can be displayed in the following rectangular co-ordinate system.



The shaded portion is XY and MNOP is the required rectangular system of XY.

Properties

The important properties of a Cartesian product are as follows:

- X.Y and Y.X have the same number of elements but X.Y $\neq$ Y.X, unless X = Y. Thus the Cartesian product of two sets is commutative if the two sets are equal.
- In the product of sets Y.X, the first component of ordered pairs are taken from Y and the second from X.
- If X and Y are disjoint sets, then X.Y and Y.X are also disjoint.
- If the set X consists of *m* elements $x_1, x_2, \ldots\ldots x_m$ and set Y consists of the *n* elements $y_1, y_2, y_3, \ldots\ldots y_n$ then the product sets X.Y consists of *mn* elements.
- If either X or Y is null then the set X.Y is also a null set.
- If either X or Y is infinite and the other is a non-empty set, then X.Y is also an infinite set.
- If $X \subset Y$, then $X.Z \subset Y.Z$
- If $X \subset Y$ and $Z \subset D$, then $X.Z \subset Y.D$.
- If $X \subseteq Y$ then $X.Y \Rightarrow (X.Y) \cap (Y.X)$
- If X, Y and Z be any three sets, then $X.(Y \cap Z) = (X.Y) \cap (X.Z)$
- If X, Y and Z be any three sets, then $X.(Y \cup Z) = (X.Y) \cup (X.Z)$
- $(X.Y) \cap (Z.D) = (X \cap Z) \times (Y \cap D)$.

The following examples contain some model applications of set theory.

Example-1:

Let A = {a, b}, B = {2, 3} and C = {3, 4}. Find (i) $A \times (B \cup C)$; (ii) $(A \times B) \cup (A \times C)$; (iii) $A \times (B \cap C)$; (iv) $(A \times B) \cap (A \times C)$.

Solution:

(i) $A \times (B \cup C) = (a, b) \times (2, 3, 4)$
$= \{(a,2), (a,3), (a,4), (b,2), (b,3), (b,4)\}$

(ii) $(A \times B) \cup (A \times C)$
$(A \times B) = \{(a, b) \times (2, 3)\} = \{(a,2), (a,3), (b,2), (b,3)\}$
$(A \times C) = \{(a, b) \times (3, 4)\} = \{(a,3)\ (a,4)(b,3)(b,4)\}$.
$(A \times B) \cup (A \times C) = \{(a,2), (a,3), (a,4), (b,2), (b,3), (b,4)\}$

(iii) $A \times (B \cap C) = \{(a, b) \times (3)\} = \{(a, 3), (b, 3)\}$

(iv) $(A \times B) \cap (A \times C) = \{(a, 3), (b, 3)\}$.

Example-2:

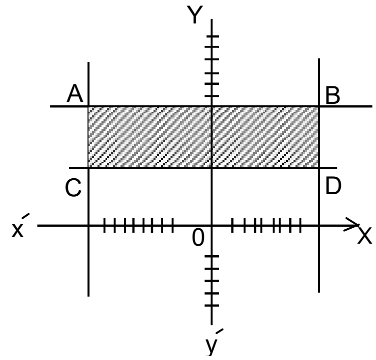
Let R represent the set of all rational numbers and
$X = \{x : x \in R \text{ and } -4 \leq x < 3.5\}$
$Y = \{y : y \in R \text{ and } 1.5 < y \leq 4.37\}$
(i) Express $X \cup Y$ and $X \cap Y$.
(ii) Draw a rectangular coordinate system and show XY on it.

Solution:

(i) $X \cup Y = \{m : m \in R \text{ and } -4 \leq m \leq 4.37\}$
$X \cap Y = \{m : m \in R \text{ and } 1.5 < m < 3.5\}$

(ii) $X.Y = \{(x,y) : x \in X, y \in Y, -4 \leq x < 3.5 \text{ and } 1.5 < y \leq 4.37\}$

The following rectangular of coordinates shows X.Y in set notation.



So, the ABCD is the required rectangular coordinate system of X.Y.

Example-3:

Let R represent the set of all real number and.

X = {x | x∈ R and –1 ≤x< 2}

Y = {y | y∈ R and 0 ≤y≤ 3}

(i) Draw a rectangular coordinate system and show X.Y on it.

(ii) Draw another rectangular coordinate system and show Y.X on it.

Solution:

(i) Element of X set = {–1, 0, 1}

Element of Y set = {0, 1, 2, 3}

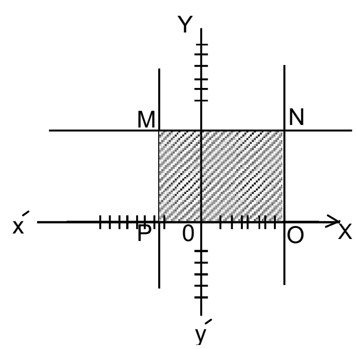
In set notation:

X.Y = {(-1,0), (-1,1), (-1,2), (-1,3), (0,0), (o,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3)}

In expression:

X.Y = {(x, y) | x∈X, y∈Y, –1 ≤x< 2 and 0 ≤ y 3}

The following rectangular of coordinates shows X.Y in set notation.



∴ MNOP is the required Rectangular system of X.Y.

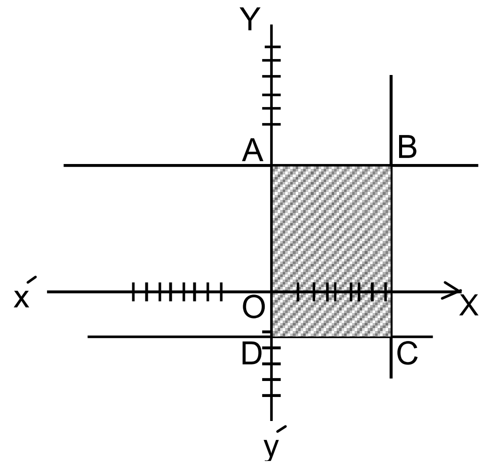
(ii) In set notation:

Y.X = {(0,1), (0,0), (0,1), (1,-1), (1,0), (1,1), (2,-1), (2,0), (2,1), (3,-1), (3,0), (3,1)}

In expression:

Y.X = {(x, y) | x∈X, y∈Y, 0 ≤ y ≤ 3 and –1 ≤x< 2}

The following rectangular coordinate shows Y.X in expression.



So, ABCD is the required rectangular co-ordinate system of Y.X.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Define the following with examples:
   Product of two sets, Difference of two sets.
2. Let the universal set, U = {a, b, c, d, e, f, g}, X = {a, b, c, d, e}
   Y = {a, c, e, g} and Z = {b, e, f, g}
   Find (i) $X \cup Z$, (ii) $Y \cap X$, (iii) Z ~Y, (iv) Y′, (v) X′ – Y, (vi) $Y' \cap Z$, (vii) (X ~ Z)′, (viii) $Z' \cap A$, (ix) (X~Y)′, (x) $(X \cap X')'$.

3. If M = {1, 2, 3}, N = {2, 3, 4}, O = {1, 3, 4} and P = {2, 4, 5}, Prove that $(M \times N)\ (O \times P) = (M \cap O) \times (N \cap P)$
4. Let R represents the set of all rational numbers and
   $X = \{x : x \in R \text{ and } -2 \le x < 3.5\}$
   $Y = \{y : y \in R \text{ and } 1.5 < y \le 4.32\}$
   (i) Express $X \cup Y$ and $X \cap Y$.
   (ii) Draw rectangular coordinate system and show (a) X.Y, and (b) Y.X on it.
5. Given A = {1, 3, 4, 7); B = {3, 7, 12}; C = {1, 5, 8}
   Write the following sets:
   (i) The set containing all elements that are members of A or members of B or members of both A & B.
   (ii) The set of elements that are members of both A and B.
   (iii) The set of elements that are members of both B and C.
   (iv) The set of elements that are members of A but not members of B.
   (v) The set of elements that are members of all three sets.

LOGARITHM

2

Unit Highlights

- Lesson – 1: Nature and Basic Laws of Logarithm
- Lesson – 2: Natural Logarithm and Antilogarithm

Technologies Used for Content Delivery

- BOUTUBE
- BOU LMS
- WebTV
- Web Radio
- Mobile Technology with MicroSD Card
- LP+ Office 365
- BTV Program
- Bangladesh Betar Program

Lesson-1: Nature and Basic Laws of Logarithm

After studying this lesson, you should be able to:

- Discuss the nature of logarithm;
- Identify the basic laws of operation of logarithm;
- Explain the characteristics and mantissa of logarithm.

Meaning of a Logarithm

Logarithm is the important tool of modern mathematics. If $a^x = n$, then x is said to be the logarithm of the number *'n'* to the base *'a'*. Symbolically it can be expressed as follows: $\log_a n = x$. In this case $a^x = n$ is an exponential form and $\log_a n = x$ is a logarithmic form. The object of logarithm is to make common calculations less laborious and the method consists in replacing multiplication by addition and division by subtraction.

Logarithm to the base *'e'* is called *'natural logarithm'* and when the base is 10, the logarithm is called *'common logarithm'*. For example,

(i) $5^3 = 125 \rightarrow \log_5 125 = 3$, i.e. the logarithm of 125 to the base 5 is equal to 3.

(ii) $(64)^{\frac{1}{6}} = 2 \rightarrow \log_2 64 = \frac{1}{6}$, i.e. the logarithm of 64 to the base 2 is equal to $\frac{1}{6}$.

Similarly,

Exponential form		Logarithmic form
$2^3 = 8$	→	$\log_2 8 = 3$
$10^2 = 100$	→	$\log_{10} 100 = 2$
$2^{-2} = \frac{1}{4}$	→	$\log_2 \frac{1}{4} = -2$
$3^0 = 1$	→	$\log_3 1 = 0$

or

Logarithmic form		Exponential form
$\log_4 64 = 3$	→	$4^3 = 64$
$\log_p R = Q$	→	$P^Q = R$
$\log_{10} 10 = 1$	→	$10^1 = 10$
$\log_5 1 = 0$	→	$5^0 = 1$

Fundamental Properties and Laws of Logarithms

The fundamental properties and laws of logarithm are as follows:

(1) The logarithm of the production of two factors is equal to the sum of their logarithms; i.e., $\log_a mn = \log_a m + \log_a n$.

(2) The logarithm of quotient is equal to logarithm of the numerator minus the logarithm of the denominator; i.e., $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$.

(3) The logarithm of any power of a number is equal to the product of the index of the power and the logarithm of the number; i.e. $\log_a m^x = x \log_a m$.

(4) Base changing formula: The formula which tells us how to change from one base to another is : $\log_b n = \frac{\log_a n}{\log_a b}$

i.e., $(\log_b n)(\log_a b) = \log_a n$

Characteristics and Mantissa of a Logarithm

The logarithm of a number consists of two parts: (i) an integer positive, negative or zero (ii) a positive or negative proper fraction. The first part is called characteristics and the second part are termed as mantissa.

Since $10^0 = 1$ $\quad\therefore \log 1 = 0$

$10^1 = 10$ $\quad\therefore \log 10 = 1$

$10^2 = 100$ $\quad\therefore \log 100 = 2$

$10^3 = 1000$ $\quad\therefore \log 1000 = 3$

$10^4 = 10{,}000$ $\quad\therefore \log 10{,}000 = 4$

Similarly, since $10^{-1} = \frac{1}{10} = 0.1,$ $\quad\therefore \log 0.1 = -1$

$10^{-2} = \frac{1}{100} = 0.01,$ $\quad\therefore \log 0.01 = -2$

$10^{-3} = \frac{1}{1000} = 0.001,$ $\quad\therefore \log 0.001 = -3$

$10^{-4} = \frac{1}{10000} = 0.0001,$ $\quad\therefore \log 0.0001 = -4.$

In general, the logarithm of a number containing n digits only in its integral part is $\{(n-1) + a\}$ fraction and the logarithm of a number having N zeros just after the decimal point is $\{-(n+1) + a\}$ fraction.

Let us take some examples on logarithm.

Example-1:

If $\log_x 625 = 4$; find the value of x.

Solution:

$\log_x 625 = 4$ can be expressed in exponential form as

$x^4 = 625$

or, $x^4 = 5^4$

or, $x = 5^{\frac{4}{4}} = 5$

Example-2:

If $\log_{\sqrt{27}} x = -\frac{4}{3}$, find the value of x.

Solution:

Expressing $\log_{\sqrt{27}} x = -\frac{4}{3}$ in the exponential form, we get $\left(\sqrt{27}\right)^{-\frac{4}{3}} = x$

or, $x = \left(\sqrt{3^3}\right)^{-\frac{4}{3}}$

or, $x = \left(3^{\frac{3}{2}}\right)^{-\frac{4}{3}}$

or, $x = 3^{-2} = \frac{1}{3^2}$

or, $x = \frac{1}{9}$

Example-3:

If $10^x = 8$, find the value of x.

Solution:

Here $10^x = 8$ can be expressed in logarithmic form as, $log_{10} 8 = x$

Therefore, $x = log_{10} 8 = 0.9030$ (by using scientific calculator).

Example-4:

The logarithm of a number is –3.153. Find the characteristics and mantissa.

Solution:

Let $log\ N = -3.153$

= (–3 – 0.153) = (–3 –1 +1 – 0.153) = – 4 + 0.847

∴ The characteristics is – 4 and mantissa is 0.847.

Example-5:

Find the logarithm whose logarithm is 2.4678.

Solution:

From the Anti-log Table,

For mantissa 0.467, the number = 2931

For mean difference 8, the number = 5

∴ For mantissa 0.4678, the number = (2931 + 5) = 2936.

The characteristics is 2, therefore the number must have 3 digits in the integral part.

Hence, antilog 2.4678 = 293.6

Example-6:

Find the number whose logarithm is – 2.4678.

Solution:

Let $log\ N = -2.4678$ = –2 –1 + 1 – 0.4678 = – 3 + .5322 = 3.5322

From Antilog Table,

For mantissa 0 .532, the number = 3404.

For mean difference 2, the number = 2

∴ For mantissa 0.5322, the number = (3404 + 2) = 3406

The characteristic is –3, therefore the number is less than one and there must be two zeros just after the decimal point.

Hence, antilog –2.4678 = 0.003406.

Example-7:

Find the value of (i) $log_2 64$; (ii) $log_3 \frac{1}{9}$; (iii) $log_9 3$ (iv) $log_8 0.25$

Solution:

(i) Let $log_2 64 = x$

$or,\ 64 = 2^x$

$or,\ 2^6 = 2^x$

$\therefore x = 6$

(ii) Let $log_3 \frac{1}{9} = x$

$or,\ \frac{1}{9} = 3^x$

$or,\ 9^{-1} = 3^x$

$or,\ 3^{-2} = 3^x$

$\therefore x = -2$

(iii) Let $log_9 3 = x$

$or\ 3 = 9^x$

$or\ 3^1 = 3^{2x}$

$or\ 2x = 1$

$\therefore\ x = \frac{1}{2}$

(iv)Let $log_8 0.25 = x$

$or,\ 0.25 = 8^x$

$or,\ \frac{1}{4} = 2^{3x}$

$or,\ 4^{-1} = 2^{3x}$

$or,\ 2^{-2} = 2^{3x}$

$or,\ 3x = -2$

$\therefore\ x = -2/3$

Example-8:

Find the logarithm of the following to the base indicated in brackets.

(i) 27, (3); (ii) 64, (8); (iii) 1000, (10); (iv) 0.25, (2).

Solution:

(i) $27 = 3^3$
$\therefore log_3 27 = 3.$

(ii) $64 = 8^2$
$\therefore log_8 64 = 2.$

(iii) $1000 = 10^3$
$\therefore log_{10} 1000 = 3.$

(iv) $0.25 = 2^{-2}$
$\therefore log_2 0.25 = -2.$

Example-9:

Without using tables, evaluate

$$log_{10}\frac{41}{35} + log_{10} 70 - log_{10}\frac{41}{2} + 2\ log_{10}5$$

Solution:

$log_{10}\left(\frac{41}{35} \times 70 \times \frac{2}{41} \times 5^2\right)$

$= log_{10} 100$

$= log_{10} 10^2$

$= 2\ log_{10} 10 = 2$

Example-10:

Simplify $7\ log\frac{10}{9} - 2\ log\frac{25}{24} + 3\ log\frac{81}{80}$

Solution:

$7\ log\frac{10}{9} - 2\ log\frac{25}{24} + 3\ log\frac{81}{80}$

$= 7\ [log\ 10 - log\ 9] - 2[log\ 25 - log\ 24] + 3\ [log\ 81 - log\ 80]$

$= 7[(log\ 5 + log\ 2) - log\ 3^2] - 2[log\ 5^2 - (log\ 3 + log\ 2^3)] + 3[log\ 3^4 - (log5 + log2^4)]$

$= 7\ log\ 5 + 7\ log\ 2 - 14\ log\ 3 - 4\ log\ 5 + 2\ log\ 3 + 6\ log\ 2 + 12\ log\ 3 - 3\ log\ 5 - 12\ log\ 2$

$= (7 - 4 - 3)\ log\ 5 + (2 - 14 + 12)\ log\ 3 + (7 + 6 - 12)\ log\ 2$

$= log2.$

Example-11:

Find the value of $log\frac{75}{16} - 2\ log\frac{5}{9} + log\frac{32}{243}$, when 10 is the base of each logarithm.

Solution:

$log\frac{75}{16} - 2\ log\ \frac{5}{9} + log\frac{32}{243}$

$= [log_{10} 75 - log_{10} 16] - 2[log_{10} 5 - log_{10} 9] + [log_{10} 32 - log_{10} 243]$

$= [(\log_{10} 5^2 + \log_{10} 3) - \log_{10} 4^2] - 2[\log_{10} 5 - \log_{10} 3^2] + [(\log_{10} 4^2 + \log_{10} 2) - \log_{10} 3^5]$
$= 2 \log_{10} 5 + \log_{10} 3 - 2 \log_{10} 4 - 2 \log_{10} 5 + 4 \log_{10} 3 + 2 \log_{10} 4 + \log_{10} 2 - 5 \log_{10} 3$
$= \log_{10} 2.$

Example-12:

Prove that,

$$(\log \frac{3}{2}).(\log \frac{4}{3}).(\log \frac{5}{4}).(\log \frac{6}{5}).(\log \frac{7}{6}).(\log \frac{8}{7}) = 3$$

Solution:

$$L.H.S. = \frac{\log 3 \times \log 4 \times \log 5 \times \log 6 \times \log 7 \times \log 8}{\log 2 \times \log 3 \times \log 4 \times \log 5 \times \log 6 \times \log 7}$$

$$= \frac{\log 8}{\log 2} = \frac{\log 2^3}{\log 2} = \frac{3 \log 2}{\log 2} = 3$$

Therefore, *L.H.S = R.H.S.* (Proved).

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Define logarithm. Is there any distinction between natural and common logarithm?
2. What are the fundamental rules of logarithmic operations?
3. Find the value of $\log_{10} 20 + \log_{10} 30 - \frac{1}{2} \log_{10} 36$
4. If $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4717$; find (i) $\log_{10} 25$; and (ii) $\log_{10} 4.5$
5. If $\log_{\sqrt{8}} x = 3\frac{1}{3}$; find the value of x.
6. Evaluate $\log \frac{31}{21} + \log 49 - \log 62 + \log 27 - \log_{30} 87$
7. If log a = 0.589; log b = 2.856 and log c = 1.963; find the value of $\log\left(\frac{a^4 b^{\frac{1}{3}}}{c^2}\right)$
8. Find the value of $\frac{1}{3} \log_{10} 125 - 2 \log_{10} 4 + \log_{10} 32.$
9. If $\log 3 = 0.4771$; $\log 2 = 0.3010$ and $\log 7 = 0.8451$, find the value of $\log \frac{48}{91}$
10. If $\log_{10}[98 + \sqrt{x^2 - 12x + 36}] = 2,$ find the value of x.
11. Show that $\log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80} = 1.$
12. Solve $\log_{10} (7x - 9)^3 + \log_{10} (3x - 4)^3 = 3.$
13. Prove that $11 \log \frac{10}{9} - 3 \log \frac{25}{24} + 5 \log \frac{81}{80} = \log 3$

Lesson-2: Natural Logarithm and Antilogarithm

After studying this lesson, you should be able to

- Explain the natural logarithm;
- Explain antilogarithm;
- Apply the principles of logarithm to solve the mathematical problems.

Nature of Natural Logarithm

Logarithms to the base *'e'* are known as *natural logarithms*. The value of *'e'* may be calculated from the *'e' series*, where

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}$ .... ∞

[Here ! is factorial, where n! = n(n-1) (n-2) .... 0!]

Hence $4! = 4 \times 3 \times 2 \times 1 \times 0!$ (since $0! = 1$)

Again, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 0!$

$= 6 \times 5!$

From *'e' series*, the value of *'e'* is 2.71828.

Let, $e^x = N$

or, $log_e N = x$

When the base of logarithm is *'e'*, it may be expressed is *ln;*

i.e., $log_e N = ln\ N$.

Again, $log_{10} N = \frac{log_e N}{log_e 10}$ *(through change of base)*

or, $log\ N = \frac{InN}{In10}$

$\therefore In\ N = log\ N \times ln\ 10$

Again, $In\ 10 = \frac{\log 10}{\log e} = \frac{1}{\log e}$

$\therefore$ In N = log N $\times \frac{1}{\log e}$

or, $log\ N = In\ N \times log\ e$

Using scientific calculator we can easily find the value of *'e'* based number:

For example, $log_e 5 = 1.6094$

$log_e 0.5 = -0.6931$

$log_e 10 = 2.3025$

$log_e e = In\ e = 1$.

Let us take same examples.

Example-1:

Find the value of *n*, if $(1.08)^n = 3$.

Solution:

Given, $(1.08)^n = 3$

or, $In(1.08)^n = In\ 3$

or, $n\ In\ (1.08) = In\ 3$

$n = \frac{\ln 3}{\ln(1.08)} = \frac{1.0986}{0.07696} = 14.27\ (App.)$

Example-2:

Find the value of i, if $(1+i)^{12} = 2$

Solution:

Here $(1+i)^{12} = 2$

or, $In\ (1+i)^{12} = In\ 2$

or, $12\ In\ (1+i) = In\ 2$

or, In (1+i) = $\frac{\ln 2}{12} = \frac{0.6931}{12} = 0.0577$

or, $(1+i) = e^{0.0577} = 1.0594$

or, $i = 1.0594 - 1 = 0.0594$

$\therefore i = 0.0594$

Anti-logarithm

Let $log_a N = x$, then N is called the anti-logarithm of x to the base a and is written in short as $antilog_a\, x$.

If $log_a N = x$, then $N = antilog_a x$

For example, if $log\ 1000 = 3$, then $antilog\ 3 = 1000$

If $log\ 708 = 2.8500$, then $antilog\ 2.8500 = 708.$

Example-3:

Find the number whose logarithm is 1.7238

Solution:

Let the number is x

Therefore, $log\ x = 1.7238$

or, x = antilog 1.7238

$\therefore\ x = 52.9420$ (by using calculator).

Example-4:

Find the value of (539.45 × 49.638)

Solution:

Let $x = 539.45 \times 49.638$

$log\ x = log\ (539.45 \times 49.638)$

$= log\ 539.45 + log\ 49.638$

$= 2.3195 + 1.6981$

or, $log\ x = 4.4276$

$\therefore x = Antilog\ 4.4276 = 26{,}776.88$

Example-5:

Solve the equation $3^x \,.\, 7^{2x+1} = 11^{x+5}$

Solution:

Taking logarithm of both sides, we have

$x\ log\ 3 + (2x+1)\ log\ 7 = (x+5)\ log\ 11$

or, $x\ log\ 3 + 2x\ log\ 7 + log\ 7 = x\ log\ 11 + 5\ log\ 11$

or, $x\ log3 + 2x\ log\ 7 - x\ log11 = 5\ log\ 11 - log\ 7$

or, $x\ (log3 + 2log7 - log\ 11) = 5\ log\ 11 - log7$

$$\therefore x = \frac{5\log 11 - \log 7}{\log 3 + 2\log 7 - \log 11} = \frac{5.2070 - 0.8451}{0.4771 + 1.6902 - 1.0414}$$

$$= \frac{4.3619}{1.1259} = 3.87\ (App.)$$

Example-6:

Find the value of $\frac{(435)^3.(0.056)^{\frac{1}{2}}}{(380)^4}$

Solution:

Let $x = \frac{(435)^3.(0.056)^{\frac{1}{2}}}{(380)^4}$

Taking logarithm of both sides, we have

$log\ x = 3\ log\ 435 + \frac{1}{2}\ log\ 0.056 - 4\ log\ 380$

or, $log\ x = 3 \times 2.6385 + \frac{1}{2} \times (-1.2518) - 4 \times 2.5798$

or, $log\ x = 7.9155 - 0.6259 - 10.3192$

or, $log\ x = -3.0296$

Hence, $x = antilog\ (-3.0296) = 0.0009341.$

Example-7:

Find the 7th root of 0.00001427

Solution:

Let x = 0.00001427

Taking logarithm of both sides, we have

$log\ x = \frac{1}{7}\ log\ (0.00001427)$

or, $log\ x = \frac{1}{7}(-4.8456)$

or, $log\ x = -0.6922$

or, $x = antilog\ (-0.6922) = 0.2031$ *(App.)*

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Find the value of $(431.96)^{26}$.
2. Find the value of $\sqrt[6]{5896.31}$.
3. Find the value of $log_2\sqrt{6} + log_2\sqrt{2/3} - log\ 10.$
4. Find the value of $log_2\sqrt{3/2} + log_2\sqrt{5/3} - log_2\sqrt{5}.$
5. Find the value of x; if $log_4 x + log_2 x = 6.$
6. Evaluate $\frac{1002.76}{12 \times 82}$ by using logarithm.
7. Solve $10^{4x-5}.32^x = 5^{3-x}.7^x$
8. Solve for x, if $log_x (8x - 3) - log_x 4 = 2.$
9. Evaluate $\frac{61.42 \times 10.70}{401.53}$

MATHEMATICS OF FINANCE

3

Unit Highlights

- Lesson – 1: Interest
- Lesson – 2: Depreciation
- Lesson – 3: Annuity

Technologies Used for Content Delivery

- BOUTUBE
- BOU LMS
- WebTV
- Web Radio
- Mobile Technology with MicroSD Card
- LP+ Office 365
- BTV Program
- Bangladesh Betar Program

Lesson-1: Interest

After studying this lesson, you should be able to

- State the nature of interest;
- Calculate the simple interest;
- Calculate the compound interest in various situations.

Nature of Interest

When *x* borrows money from *y,* then *x* has to pay certain amount to *y* for the use of the money. The amount paid by *x* is called interest. The amount borrowed by *x* from *y* is called principal. The sum of the interest and principal is usually called the total amount. When interest is payable on the principal only, it is termed as simple interest. On the other hand, when interest is calculated on the amount of the previous year or period, then it is called compound interest.

Calculation of Simple Interest

Let P= Principal i.e., the initial sum of money invested.

I = Interest per unit money/ per unit time.

X = Period i.e., unit of time for which the interest is calculated.

A = Amount i.e., principal plus interest accrued.

The interest on 1 unit of money for 1 unit of time = i

The interest on 1 unit of money for 'n' unit of time = ni

The interest on P unit of money for 'n' unit of time = Pni

Hence A= P + Pni = P (1+ni)

The simple interest obtained on principal (P) after n years will be

= A - P

= P (1+ni) – P = (P + Pni – P) = Pni

For example, the rate of simple interest is 10% per annum means that the interest payable on Tk.100 for one year is Tk.10, i.e., at the end of one year, total amount will be Tk.110, at the end of second year, it will be Tk.120 and so on.

Example-1:

Mr. Rahim has invested Tk.30,000 for 5 years at 10% rate of interest. What will be the simple interest and amount after 5 years?

Solution:

We know that the simple interest on principal (P) for 'n' year at a rate 'i' = Pni

Here P= 30,000 , N= 5 , i= 10%=0.10

Substituting the given values we have,

Simple Interest = 30,000 × 5 × 0.10

= Tk.15,000

Hence the required simple interest of 5 years is Tk.15,000

Amount after 5 years at simple interest, A = P (1+ni)

= 30,000 (1 + 5 × 0.10)

= 30,000 (1.50)

= Tk.45,000

Calculation of Compound Interest

If *i* be the rate of interest per unit per period, a principal 1 accumulates at compound interest in the following manner. At the end of every period, the interest earned is added to the principal to become the principal earning interest for the next period, For example

		Amount
Principal (P)	*1*	1
Interest for the first period	*i*	
Principal for the 2^{nd} period	*(i+i)*	(1+i)
Interest for the 2^{nd} period	*i (1+i)*	

Principal for the 3^{rd} period *(1+i) (1+i)* $= (1+i)^2$
Interest for the 3^{rd} period $i(1+i)^2$
Principal for the 4^{th} period $(1+i)^2 (1+i)$ $= (1+i)^3$
And so on.
Hence the amount at the end of *n* period = $(1+i)^n$
Thus the amount (A) of Principal (P) at the end of *n* periods is,

$$A = P (1+i)^n$$

The fundamental formula of compound interest, namely $A = P (1+i)^n$ is easily adopted to logarithmic calculation, where

$$\log A = \log P + n \log (1+i)$$

Now, the compound interest $= A - P$
$= P (1+i)^n - P$
$= P [(1+i)^n - 1]$

Let *P* = Principal, *A* = the total amount, *t* = total interest, *i* = annual rate of interest, *n* = number of period; then the compound interest can be computed by suing the following formula, which may be changed on the basis of the number of compounding time.

Compounding Time	**Total amount**	**I = Total amount (A) – Principal amount (P)**
Weekly	$A= P (1 + \frac{i}{52})^{52n}$	$I = P [(1 + \frac{i}{52})^{52n} - 1]$
Monthly	$A = P (1 + \frac{i}{12})^{12n}$	$I = P [(1 + \frac{i}{12})^{12n} - 1]$
Compounding Time	**Total amount**	**I = Total amount (A) – Principal amount (P)**
Quarterly	$A = P (1 + \frac{i}{4})^{4n}$	$I = P [(1 + \frac{i}{4})^{4n} - 1]$
Half yearly	$A= P (1 + \frac{i}{2})^{2n}$	$I = P [(1 + \frac{i}{2})^{2n} - 1]$
Annually	$A= P (1 + i)^n$	$I = P[(1 + i)^n - 1]$

If the interest is *i* per unit per annum, nominal convertible *'m'* times a year; i/m is converted into the principal at the end of every such compounding time; and 1 will accumulate to $(1+i/m)^m$ in a year. The difference $[(1+i/m)^m - 1]$ a year on a principal 1 is known as the effective rate of interest per annum.

A Principal *m* accumulates to $A = (1+i/m)^{nm}$ in n year at the above rate.

Example-2:

Mr. Rahim has invested Tk.30,000 for 4 years at 12% rate of interest.

1. What will be the compound interest and amount after 4 years if it is compounding (a) Yearly; or (b) Monthly?
2. Find the number of years in which the sum will double itself at annual compound interest.
3. What should be the annual compound interest rate to make the amount Tk.60,000 after 4 years?

Solution:

We are given, $P = 30,000$, $n = 4$ and $i = 0.12$

(1)

(a) In the case of Yearly Compounding:

Compound interest after 4 years = $P [(1+i)^n - 1]$
$= 30,000 [(1 + 0.12)^4 - 1]$
$= 30,000 [(1.12)^4 - 1]$
$= 30,000 \times 0.5735 = Tk.17,205$

Amount after 4 years, $A = P\ (1+i)^4$

$= 30{,}000\ (1 + 0.12)^4$

$= 30{,}000 \times 1.5735$

= Tk.47, 205

(b) In the case of Monthly Compounding: [then $m = 12$]

Compound interest after 4 years = $P\ [(1+{}^{i}/_{m})^{mn} - 1]$

$= 30{,}000\ [(1 + \left(1+\frac{0.12}{12}\right)^{12\times4} - 1]$

$= 30{,}000[1.6122 - 1]$

$= 30{,}000 \times 0.6122$ = Tk.18, 366

Amount after 4 years, A = $P\ (1+{}^{i}/_{m})^{mn}$

$= 30{,}000\ \left(1+\frac{0.12}{12}\right)^{12\times4}$

$= 30{,}000\ (1.01)^{48}$

$= 30{,}000 \times 1.6122$

= Tk.48, 366

2. Let the sum will be Tk.(30,000 × 2) = Tk.60, 000 is n years.

So, $P = 30{,}000$, $A = 60{,}000$, $i = .12$ and n =?

Now, $A = P\ (1+i)^n$

Or, $60{,}000 = 30{,}000\ (1 + 0.12)^n$

Or, $(1.12)^n = 60{,}000/30{,}000$

Or, $(1.12)^n = 2$

Taking logarithm both sides, we have

or, n log1.12 = log2

$n = \frac{\log 2}{\log 1.12} = \frac{0.3010}{0.0492} = 6.12$ years.

Hence it will take 6.12 years for Tk.30, 000 to be doubled to Tk.60,000.

3. We have, $P = 30{,}000$; $A = 60{,}000$, $n = 4$ and i =?

Now $A = P\ (1+i)^n$

or, $60{,}000 = 30{,}000(1+i)^4$

or, $(1+i)^4 = 60{,}000/30{,}000$

or, $(1+i)^4 = 2$

Taking logarithm both sides, we have

or, $4\log(1+i) = \log 2$

or, $\log(1+i) = \frac{\log 2}{4} = \frac{0.3010}{4}$

or, $\log(1+i) = 0.0753$

or, $(1+i)$ = antilog 0.0753

or, $(1+i) = 1.1893$

or, $i = (1.1893 - 1) = 0.1893$ or 18.93%

Hence the rate of interest should be 18.93% to make the amount Tk.60,000 after 4 years.

Calculation of compound interest with growing investment (withdrawals)

Let Ainvested at the beginning of the first year and an additional sum B be added to the investment in each subsequent year. No withdrawals are to be made and whose sum invested is to be allowed to accumulate at a compound rate.

Hence $A = (A_0+B/i)\ (1+i)^n - B/i$

Here, A = the sum of amount

i = the rate of interest

A_0 = invested at the beginning of the year.
B = additional sum to be added / (withdrawn) in investment.
n = number of periods.

Therefore compound interest = Total Amount (A) – Principal Amount.

Let us illustrate it by the following examples.

Example – 3:

Tk.10,000 is invested at the beginning of 1999. It remains invested and, on 1st January in each subsequent year, another Tk.500 is added to it. What sum will be available on 1st January 2005 if interest is compounded each year at the rate of 5% per annum?

Solution:

We know that, $A = (A_0+B/i)\ (1+i)^n - B/i$

Here, $A_0 = 10{,}000$, B = 500, $i = .05$, $n = 6$

Substituting the values we have,

$A = (10{,}000 + 500/0.05)\ (1+ .05)^6 - 500/0.05$
$= (10{,}000 + 10{,}000)\ (1.05)^6 - 10{,}000$
$= 20{,}000 \times 1.3401 - 10{,}000$
$= 26{,}802 - 10{,}000 =$ Tk.16, 802

Therefore, the sum of amount on 1st January 2005 is Tk.16, 802.

Example–4:

A man invests Tk.10,000 once at how and withdraws Tk.1500 at the end of each year starting at the end of the first year. How much will have left after seven years if the money is invested at 4% per annum?

Solution:

We know that, $A = (A_0 + B/i)\ (1+i)^n - B/i$

Here, $A_0 = 10{,}000$, $n = 7$, $i = 0.04$, B = - 1,500

Substituting the values we have

$A = \{10{,}000 + (-1500)/0.04\}\ (1+ 0.04)^7 - (-1500/0.04$
$= (10{,}000 - 37{,}500)\ (1.04)^7 + 37{,}500$
$= (-27{,}500)\ (1.3159) + 37{,}500$
$= - 36187.25 + 37{,}500$
= Tk.1, 312.75

Drawing at this rate he will only have Tk.1, 312.75 at the end of seven years.

Questions For Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Define the following: Simple interest, Compound interest, Effective rate of interest.
2. Compare between simple interest and compound interest.
3. Mr. Asif has invested Tk.1,00,000 for 5 years at 10% rate of interest.
   a. What will be the simple interest and amount after 5 years?
   b. What will be the compound interest and amount after 5 years if interest is paid
   (i) Monthly, or (ii) Quarterly?
   c. What should be annual compound interest rate to make the amount Tk.2,00,000 after 5 years?
4. At what rate of interest an amount of investment will be thrice as much as at the end of 6 years?
5. How many years will it take at 12% interest compounding annually for Tk.6000 to grow to Tk.11, 000?
6. Tk.50,000 invested at the beginning of 1997. It remains invested and, on 1st January of each subsequent year, another Tk.5000 is added to it. What sum will be available on 1st January 2005 if interest is compounded yearly @ 10% per annum?

Lesson-2: Depreciation

After studying this lesson, you should be able to:

- Explain the nature of depreciation and depreciated value;
- Calculate the amount of depreciation under the different methods of depreciation.

Nature of Depreciation

In case of depreciation, the principal value is diminished every year by some amount, and in the subsequent period the diminished value becomes the principal value. In case of uniform decrease or depreciation, '*i*' is to be substituted by '*–i*' in the formula of future value. In that case depreciated value and accumulated depreciation is calculated by using the following formula:

Depreciated value = $P\ (1-i)^n$[Under reducing balance method]

Accumulated depreciation = $P\ [1-(1+i)^n]$

Where, P = Cost price of the asset

i = Rate of depreciation

n = Number of periods the asset has been depreciated.

For calculation of depreciated value and accumulated depreciation the following examples are highlighted here.

Example-1:

A machine has been purchased in 1999 at a cost of Tk.3,00,000. The machine is depreciated @8% per annum on reducing balance method. Compute-

i. What would be the depreciated value of the machine at the end of 2005?
ii. What amount should be charged as depreciation of the machine for 2006?
iii. Would it be profitable to sale the machine for Tk.1,20,000 at the end of 2007?
iv. When the depreciated value of the machine will be Tk.1,02,550?

Solution:

We are given, P = 3,00,000, i = 0.08

i. The machine has been purchased in 1999. At the end of 2005, it will be 7 years' old. Hence, the depreciated value of the machine at the end of 2005 would be,

$= P\ (1-i)^n$

$= 3{,}00{,}000\ (1-0.08)^7 = (3{,}00{,}000 \times 0.5578) = \text{Tk.}1{,}67{,}340.$

ii. The depreciated value of the machine at the end of 2006, would be,

$= P\ (1-i)^n$

$= 3{,}00{,}000\ (1-0.08)^8$

$= (3{,}00{,}000 \times 0.5132) = \text{Tk.}1{,}53{,}960$

Therefore depreciation for 2006 would be:

(1,67,340 – 1,53,960) = Tk.1,41,630

iii. The depreciated value of the machine at the end of 2007 would be,

$= P\ (1-i)^n$

$= 3{,}00{,}000\ (1-0.08)^9$

$= 3{,}00{,}000 \times 0.4721 = \text{Tk.}1{,}41{,}630$

Hence, it would not be profitable to sale the machine for Tk.1,20,000 at the end of 2007.

iv. Let after n years the depreciated value of the machine would be Tk.1,02,550.

Now $3{,}00{,}000\ (1-0.08)^n = 1{,}02{,}550$

or, $(0.92)^n = 1{,}02{,}000/3{,}00{,}000 = 0.3418$

Taking logarithm both sides we have

or, $n \log 0.92 = \log 0.3418$

So, $n = \dfrac{\log 0.3418}{\log 0.92} = \dfrac{-0.4662}{-0.0362} = 12.88$ years.

Therefore, after 12.88 years the depreciated value of the machine would be Tk.1,02,550.

Different Methods for Calculation of Depreciation

The depreciation calculations have a significant impact on cash flows after taxes (CFAT). It is because a firm can legitimately deduct depreciation from its gross income to arrive at it's before tax income. Different methods of depreciation affect tax liability, and hence the cash flows differently. Generally, there are three methods of depreciation calculations which are discussed as under.

(1) **Straight Line Method:** Under this method, depreciation charges are allocated equally over the asset's economic life. The amount of annual depreciation charge is given by the formula:

Amount of Annual Depreciation = (Original cost – Salvage Value) ÷Economic life of an asset.

(2) **Sum-of-the-year's-Digits Method:** In this method, the depreciation base in each year is the same as in the straight-line method - *original cost less salvage value.* However, depreciation factor changes in each year. The depreciation factor for any year is the number of useful years remaining in the life of the project taken from the beginning of the year divided by the sum of a series of numbers representing the years of service life. The depreciation factor of multiplier is calculated by the following formula:

Multiplier = $\frac{n(n+1)}{2}$ where n = Economic life of asset.

(3) **Double Declining Balance Method:** It is more popularly known as the twice straight line depreciation method. Under this method, the amount of depreciation to be charged is twice the straight line rate. For example, a machine which has been purchased for Tk.2,20,000 has a salvage value of Tk.20,000 and economic life of 5 years. The straight line depreciation would be Tk.40,000 per year and, therefore, annual depreciation will be $\left(\frac{40{,}000}{220000-20000}\right)$ = 20% of the depreciable value. Then the rate would be 40% under the double declining balance method. This 40% depreciation rate would be applied to the book value of the asset each year until book value equals salvage value (Tk.20,000). Here, annual depreciation = $\left[\frac{Book\ value}{Economic\ life}\times 2\right]$.

Let us take an example to illustrate the depreciation calculations under different methods of depreciation.

Example-2:

A firm purchased a machine for Tk.4,00,000. Its useful life was 5 years and salvage value of Tk.10,000. Calculate the amount of depreciation under different methods of depreciation.

(1) Straight Line Depreciation Method

Depreciation Annual = (4,00,000-10,000)/5 = 3,90,000/5 = 78,000 (From 1st year to 5th Year)

(2) Sum-of-the-Years Digit Depreciation Method.

Depreciation factor/Multiplier (S) = $\frac{n(n+1)}{2} = \frac{5(5+1)}{2}$ = 5 × 3 = 15

Year	Multiplier	Depreciation (in Tk.)	
1	5/15	($^5/_{15}$ × 3,90,000) =	1,30,000
2	4/15	($^4/_{15}$ × 3,90,000) =	1,04,000
3	3/15	($^3/_{15}$ × 3,90,000) =	78,000
4	2/15	($^2/_{15}$ × 3,90,000) =	52,000
5	1/15	($^1/_{15}$ × 3,90,000) =	26,000

(3) Double Declining (reducing) Balance Depreciation Method.

Year	Total Investment	Depreciation (in Tk.)	Year-end Book value
1	3,90,000	(3,90,000/5) × 2 = 1,56,000	2,34,000
2	2,34,000	(2,34,000/5) × 2 = 93,600	1,40,400
3	1,40,400	(1,40,400/5) × 2 = 56160	84240
Year	Total Investment	Depreciation (in Tk.)	Year-end Book value
4	84,240	(84,240/5) × 2 = 33,696	50,544
5	50,544	= 50544	00

Example-3:

A machine depreciates @ 10% of its value at the beginning of the year. The machine was purchased for Tk.5,810 and the scrap value realized when sold was Tk.2,250. Find out the number of years during which the machine was in use.

Solution:

We know, $A = P(1-i)^n$

Here A = 2,250, P = 5,810, i = 0.10, n =?

Substituting the given value we have

$$2{,}250 = 5{,}810\,(1-i)^n$$

$$\text{or, } (1 - 0.10)^n = \frac{2250}{5810}$$

$$\text{or, } (0.90)^n = 0.38726$$

Taking logarithm both sides we get

$$n \log 0.90 = \log 0.38726$$

$$\text{or, } n(-0.04576) = -0.412$$

$$\text{or } n = \frac{-0.412}{-0.04576} = 9$$

Questions for review

1. A machine has been purchased in 1995 at a cost of Tk.1,00,000. The machine is depreciated @12% p.a. on reducing balance method.
   A. What would be depreciated value of the machine at the end of 2005?
   B. What amount should be charged as depreciation of the machine for 2001?
   C. Would it be profitable to sale the machine for Tk.5,00,000 at the end of 2002?
   D. When the depreciated value of the machine will be Tk.4,20,550?
2. The value of a machine depreciates @ 10% p.a. If its present value is Tk.81000, what will be its worth after 2 years? What was the value of the machine 2 years ago?
3. A machine depreciates @ 12% p.a of its value at the beginning of a year. The machine was purchased for Tk.58,100 and the scrape value released when sold was Tk.10,000 Find out the number of years during which the machine was in use?

Lesson-3: Annuity

After studying this lesson, you should be able to:

- Explain the different types of annuities;
- Calculate annuity by yourself.

Nature of Annuity

A series of uniform payments is called an annuity. In other words, an annuity is a series of payments of a fixed amount at regular intervals generally. The interval is a year, but it may be six months, or a quarter or a month.

Annuities can be divided into two classes – (1) Annuity certain and (2) Annuity contingent. In annuity certain, the payments are to be made unconditionally, for a certain or fixed number of years. In annuity contingent, the payments are to be made till the happening of some contingent even such as the death of a person, the marriage of a girl, the education of a child reaching a specified age. Life annuity is an example of annuity contingent.

Annuity certain can be divided into :(i) annuity due; and (ii) immediate annuity. When the payment of an annuity is at the beginning of each period, it is said to be an annuity due. When the payment is at the end of each period, the annuity is termed as immediate annuity.

The Present Value of an Annuity

The present value of an annuity is the sum of the present values of its installments. In calculating the present value of an annuity it is always customary to reckon compound interest.

Let A be the annuity, V is the present value, i is the rate of interest per year and n the number of years to continue, and then the present value of an immediate annuity is calculated by the following formula:

$$V = \frac{A}{i}\left[1 - \frac{1}{(1+i)^n}\right]$$

On the other hand the present value of an annuity due is calculated by the following formula:

$$V = \frac{A}{i}(1+i)\left[1 - \frac{1}{(1+i)^n}\right]$$

or, $V = A/i\ (1+i)\ [1 - 1/(1+i)^{n-1}]$

Let us illustrate it by two examples.

Example-1:

An investment will yield Tk.10,000 per annum for 8 years. If finance can be obtained at 7% per annum and the investment costs Tk.50,000, is it worth undertaking?

Solution:

We know that the present value of the immediate annuity would be

$V = A/i\ [1 – 1/(1+i)^n]$

Here A = Tk.10,000

$i = 0.07$

$n = 8$

Substituting the given values we have

$V = 10{,}000/0.07\ [1–1/(1+0.07)^8]$

$= 10{,}000/0.07\ [1–1(1.07)^8]$

$= 1{,}42{,}857\ [1–1/1.7182]$

$= 1{,}42{,}857\ [1– 0.5820]$

$= 1,42,857 \times 0.4180$
$= \text{Tk.}59,714.23$ (App)

Since the investment's actual cost is Tk.50,000 and the present value of the annuity is Tk.59,714.23; the investment should be made.

Example-2:

Mr. Karim can purchase a machine by paying Tk.40,000 in cash at now. He can also purchase the machine by 8 equals' yearly installments to be paid at the beginning of each year. If the interest rate is 12%, what should be amount of each installment?

Solution

Let A be the annual installment. Then Tk.40,000 is the present value of this annuity due. We are given,

$V = 40,000$, $n = 8$, $i = 0.12$ and $A = ?$

Using the formula

$$V = \frac{A(1+i)}{i}\left[1 - \frac{1}{(1+i)^n}\right]$$

$$\text{Or, } 40,000 = \frac{A(1+0.12)}{0.12}\left[1 - \frac{1}{(1+0.12)^8}\right]$$

$$\text{Or, } 40,000 = A\left[\frac{1.12}{0.12}\right]\left[1 - \frac{1}{2.4760}\right]$$

Or, 40,000 = A (9.3333) (1-0.4039)
Or, 40,000 = A (9.3333) (0.5961)
Or, 40,000 = A (5.5636)
Or, A = ($^{40000}/_{5.5636}$) = 7189.59

Hence the amount of each investment should be Tk.7189.59

Amount of an Annuity

Let A be the annuity, i the rate of interest per year, n the total time period of an annuity and M the future amount of annuity after n years, then the total amount of an immediate annuity is calculated by the following formula:

$$M = \frac{A}{i}\left[(1+i)^n - 1\right]$$

On the other hand, the total amount of an annuity due is calculated by the following formula:

$$M = \frac{A(1+i)}{i}\left[(1+i)^n - 1\right]$$

In the repayment of a loan, it is sometimes arranged that the repayment is to be made in equal periodical installments, including repayment of principal and interest. The difference between amount of installment and interest is termed as amortization.

The following section of this lesson contains some model applications of amount of an annuity.

Example-3:

A machine costs the company Tk.98,000 and its effective life is estimated to be 12 years. If the scrap realizes Tk.3,000 only, what amount should be retained out of profits at the end of each year to accumulate at compound interest at 5% per annum?

Solution:

Let A be the annual installment. Evidently the amount of the annuity A to continue for 12 years, i.e. the balance amount to be retained = (98,000 - 3,000) = 95,000

We know that M = A/i [(1+i)^n -1]
Here, M = 95,000; i = 0.05; n = 12 and A= ?
Now putting the values we get,

95,000 = A/0.05 [(1+0.05)^{12} -1]
Or, 95,000 = A/0.05 [(1.05)^{12} -1]
Or, 95,000 = A/0.05 [1.7959 - 1]
Or, 95000 0.05 = A (0.7959)

$$\text{Or, } A = \frac{4750}{0.7959} = 5968.09$$

So, Tk.5968.83 should be retained out of profits at the end of each year.

Example-4:

Mr. Zahad wants to purchase a machine after 10 years when it will cost Tk.6,00,000. From now, he wants to save money for the machine and plans to deposit money into bank in 10 equal installments, the first deposit is to be made immediately. Calculate the amount of each installment reckoning compound interest at 10% p.a.

Solution:

Here the deposit pattern is an annuity due. We are given,

M = 6,00,000; i = 0.10; n = 10 and A = ?

Now using the formula

M = A/i (1+i) [(1+i)^n -1]
or, 6,00,000 = A/0.10(1+0.10) [(1+0.10)^{10} -1]
or, 6,00,000 = A/0.10(1.10) [(1.10)^{10}-1]
or, 6,00,000 = A(11) (2.5937-1)
or, 6,00,000 = A(11) (1.5937)
or, 6,00,000 = 17.5307

$$\text{or, A} = \frac{600000}{17.5307} = 34225.67$$

Hence Mr. Zahad has to deposit Tk.34,225.67 in each installment at the beginning of each year.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written from) the following questions:

1. Define the following with example:

   Annuity, Annuity certain, Annuity due, Contingent annuity, Immediate annuity and Amortization.

2. Mr. Karim buys a house worth Tk.3,50,000. The contract is that Karim will pay Tk.1,00,000 immediately and the balance in 15 annual equal installments with 10% per annum compound interest. How much he has to pay annually?
3. A man wishes to have Tk.1,50,000 available in a bank account when his daughter's first year college expenses begin. How much must he deposit now at 12% compounded annually if the girl is to start in college five years from now?
4. A man retires at the age of 55 years from active service and his employer gives him pension of Tk.1,500 a year paid in half yearly installments for the rest of his life. Assuming his expectation of remaining life to be 15 years and that interest is 12% p.a. payable half yearly? What single sum is equivalent to his pension?
5. Hena borrowed Tk.10,000 to buy a refrigerator. She will amortize the loan by monthly payment of Tk.R each over a period of 3 years. Find the monthly payment if interest is 12% compounding monthly. Also find the total amount Hena will pay.

EQUATIONS THEORY

4

Unit Highlights

- Lesson – 1: Equation and Identity
- Lesson – 2: Inequality
- Lesson – 3: Degree of an Equation
- Lesson – 4: Quadratic Equation

Technologies Used for Content Delivery

- BOUTUBE
- BOU LMS
- WebTV
- Web Radio
- Mobile Technology with MicroSD Card
- LP+ Office 365
- BTV Program
- Bangladesh Betar Program

Lesson 1: Equation and Identity

After studying this lesson, you should be able to:

- Explain the nature and characteristics of equations;
- Explain the nature and characteristics of identities;
- Solve the equations;
- Solve the inequalities.

Introduction

Many applications of mathematics involve solving equation. In this lesson we will discuss the equation, identities and uses of equations.

Equation

An equation is a statement which says that two quantities are equal to each other. An equation consists of two expressions with a '=' sign between them. In other words, if two sides of an equality are equal only for particular value of the unknown quantity or quantities involved, then the equality is called an equation.

For example, $4x = 8$ is true only for $x = 2$. Hence, it is an equation.

An equation which does not contain any variable is either a true statement, such as $2 + 3 = 5$, or a false statement, such as $3 + 5 = 12$. If an equation contains a variable, the solution set of the equation is the set of those values for the variable which gives a true statement when substituted into the equation.

For example, the solution set of $y^2 = 4$ is $(–2, 2)$, because $(–2)^2 = 4$, and $2^2 = 4$, but $y^2 \neq 4$ if y is any number other than –2 or 2.

Identities

The equations signify relation between two algebraic expressions symbolized by the sign of equality. If two sides of an equality are equal for all values of the unknown quantity or quantities involved, then the equality is called an identity.

For example, $x^2 – y^2 = (x + y) (x – y)$ is an identity.

We can prove that identities hold true for whatever are values of the variables substituted in these. It we use $x = 2$ and $y = 3$ in the above identity, we have $(2)^2 – (3)^2 (2 + 3) (2 – 3)$

$$\text{or, } 4 – 9 (5) (–1)$$
$$\text{or, } – 5 = – 5$$

Again, by substituting the values of $x = – 4$ and $y = – 6$, we have

$$(–4)^2 – (–6)^2 = (–4 – 6) (–4 + 6)$$
$$\text{or, } 16 – 36 = (–10) (2)$$
$$\text{or, } –20 = –20$$

Hence, identities hold true for whatever value is put for variables.

Derived Identities

Derived identities are the identities derived by transposing the values in the basic identities and are very useful in tackling some problems in mathematics. For example,

(1) Identity $\rightarrow (x + y)^2 = x^2 + 2xy + y^2$ .... (i)
Derived Identities $x^2 + y^2 = (x + y)^2 – 2xy$
and $2xy = (x + y)^2 – (x^2 + y^2)$

(2) Identity $\rightarrow (x – y)^2 = x^2 – 2xy – y^2$ .... (ii)
Derived Identities $x^2 + y^2 = (x – y)^2 + 2xy$
and $2xy = x^2 + y^2 – (x – y)^2$

By adding (i) and (ii)

$(x + y)^2 + (x - y)^2 = 2(x^2 + y^2)$ .... (iii)

By substituting (ii) from (i), we get

$(x + y)^2 - (x - y)^2 = 4xy$ .... (iv)

By dividing both (i) and (ii) by 4 and then subtracting (ii) from (i), we have $[(x + y)^2 / 4] - [(x - y)^2]$

The following section of this lesson contains some model applications of equations.

Example–1:

Solve, $\dfrac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} = \dfrac{1}{3}$

Solution:

$$\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} = \frac{1}{3}$$

By cross multiplying, we have

$3\sqrt{1+x} - 3\sqrt{1-x} = \sqrt{1+x} + \sqrt{1-x}$

or, $3\sqrt{1+x} - 3\sqrt{1+x} = \sqrt{1-x} + \sqrt{1-x}$

or, $2\sqrt{1+x} = 4\sqrt{1-x}$

Squaring both sides, we have,

$4(1 + x) = 16 (1 - x)$

or, $4 + 4x = 16 - 16x$

Transposing the term 16x and 4

$4x + 16x = 16 - 4$

or, $20x = 12$

or, $x = \dfrac{3}{5}$

Therefore, $x = \dfrac{3}{5}$ is the solution of the given equation.

Example – 2:

The sum of two numbers is 45 and their ratio is 7:8. Find the numbers.

Solution:

Let, one of the numbers be x

The other number is $(45 - x)$

Using the given information, we get, $\dfrac{x}{45 - x} = \dfrac{7}{8}$

By cross multiplication, we get

$8x = 7 (45 - x)$

or, $8x = 315 - 7x$

Transposing the term $-7x$, we have

$8x + 7x = 315$

or, $15x = 315$

or, $x = \dfrac{315}{15} = 21$

Hence, the one number is 21 and the other number is (45 – 21) = 24.

Example – 3:

The ages of a mother and a daughter are 31 and 7 years respectively. In how many years will the mother's age be $\frac{3}{2}$ times that of the daughter?

Solution:

Let the required number of years be *x*.
Mother's age after *x* years = 31 + x
Daughter's age after *x* years = 7 + x
Using the given information, we get

$$31 + x = \frac{3}{2}(7 + x)$$

or, $$31 + x = \frac{21 + 3x}{2}$$

By cross multiplication, we have

$$2(31 + x) = 21 + 3x$$

or, $$62 + 2x = 21 + 3x$$

Transposing the terms 3x and 62

$$2x - 3x = 21 - 62$$

or, $$-x = -41$$

or, $$x = 41$$

Hence, in 41 years, the mother's age will be $\frac{3}{2}$ times age of the daughter's.

Questions For Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. What is an equation and inequalities? Mention the characteristics of equation.
2. What is the difference between identity and equation? Give examples.
3. Solve the following equations:
   (i) x (x + 1) + 72 / x (x+1) = 18
   (ii) $x^2 - 6x + 9 = 4\sqrt{x^2 - 6x + 6}$
   (iii) $\sqrt{\frac{x}{x+6}} + \sqrt{\frac{x+6}{x}} = \frac{25}{12}$
4. Wasifa's mother is four times as old as Wasifa. After five years, her mother will be three times as old as she will be then, what are their present ages?
5. A steamer goes downstream and covers the distance between two parties in 4 hours while it covers the same distance upstream in 5 hours. If the speed of the stream is 21 cm/per hour, find the speed of the steamer in still water.
6. Three prizes are to be distributed in a quiz contest. The value of the 2^{nd} prize is five-sixths the value of the first prize and the value of the third prize is four fifths that of the second prize. If the total value of the three prizes is Tk.15,000, find the value of each prize.

Lesson-2: Inequality

After studying this lesson, you should be able to:

- Describe the nature of inequalities;
- Explain the properties of inequality;
- Solve the inequalities.

Nature of Inequality

Relationship of two expressions with an inequality sign (≤or ≥, < or >) between them is called inequality. For example,

$x > y \rightarrow$ "x is greater than y"
$x < y \rightarrow$ "x is smaller than y"
$x \ngtr y \rightarrow$ "x is not greater than y"
$x \nless y \rightarrow$ "x is not smaller than y"
$x \leq y \rightarrow$ "x is smaller than or equal to y"
$x \geq y \rightarrow$ "x is greater than or equal to y"

Properties of Inequalities

The fundamental properties of inequalities are as follows:

(a) Order Axioms: If x and b are only elements, then

(i) One and only one of the following is true:

$x = b, x < y$ and $x > y$

(ii) If $x < y$ and $y < z,$ then $y < c$

(iii) If $x < y$ and $x < z,$ then $xz<yz$

Since, '$x > y$' and '$y < x$' are the same statements, the above axioms can be replaced in terms of '$x > y$'.

As shown earlier sometimes equality signs are combined with inequality signs $x < y$ means $x = y$ or $x < y$.

Again $x < y$ means x is not less than y and that means either $x > y$.

So, $x < y$ means $y \leq x$.

We also say that x is positive when $x \geq 0$ and x is negative, when $x < 0$.

(b) Operation Axioms:

(i) All equals may be add or subtracted from both sides of inequalities and the inequality is preserved.
For example, if $5x - 9 < 12$
We may add 5 to both sides and we get
$5x - 9 + 5 < 12 + 5$
or, $5x - 4 < 17$

Any term in an inequality can be moved from one side to the other provided that its sign is changed. For example, if $5x - 4 < 17$.
or, $5x < 17 + 4$
And, again if $x - z > y$
or, $x > y + z$

(ii) Both sides of an inequality may be multiplied or divided by a positive number and the inequality is preserved. For example, if $12x < 36$.
After Multiplying the both sides of inequality by 5, we get
$12x \times 5 < 36 \times 5$
or, $60x < 180$
Again, after dividing both sides of inequality by 3, we get
$12x \div 3 < 36 \div 3$
or, $4x < 12$

(iii) Both sides on an inequality are multiplied or divided by a negative number and the direction of the inequality is reversed. For example, if $7x < 40$, (where $x = 4$).
By multiplying both sides of the inequality by –5, we have
$7x \times (-5) > 40 \times (-5)$ [Note that inequality sign has been changed from <to>]
or, $-35x > -200$
This is because when x = 4, the inequality $-35x = -140$ is greater than –200.

(iv) An inequality can by converted into an equation:
If $x > y$ then $x = y + p$
Where p is the positive real number (i.e. $p > 0$)
If $z > m$, then we write $z = m + q$, where $q > 0$
Hence, $x, z = (y + p)\ (m + q) = ym + yq + pm + pq$
Now p and q are positive. If in addition y and m are positive, then every term on the right-hand side is also positive so that
$x.\ z > y.\ m$

(v) If $\frac{x}{z} > \frac{y}{m}$, then $\frac{z}{x} < \frac{m}{y}$

(vi) If $x < y$, then $-x > -y$

(vii) Now if $x_1 > y_1,\ x_2 > y_2,\ x_3 > y_3$ ----------- $x_n > y_n$,
then $x_1 + x_2 + x_3 +$ ----------- $+x_n > y_1 + y_2 + y_3 +$ --------- $+y_n$
and $x_1.\ x_2.\ x_3$ --------- $x_n > y_1.\ y_2.\ y_3$ ----------- y_n

(viii) If $x > y$ and $n > 0$ then $x^n > y^n$ and $\frac{1}{x^n} < \frac{1}{y^n}$

The following section of this lesson contains some applications of inequality.

Example-1:

Solve: $3[4x - 5(2x - 3)] \leq 7 - 2\ [x + 3\ (4 - x)]$

Solution:

$3[4x - 5(2x - 3)] \leq 7 - 2\ [x + 3\ (4 - x)]$
or, $3[4x - 10x - 15)] \leq 7 - 2\ [x + 12 - 3x)]$
or, $12x - 30x - 45 \leq 7 - 2x + 24 + 6x$
Transposing both sides we have
or, $12x - 30x + 2x \leq 6x - 24 - 45$ or, $-12x \leq -62$
Multiplying both sides by –1, we get
or, $22x \geq 62$
or, $x \geq \frac{31}{11}$

Example-2:

Solve: $5x - 2(3x - 4) > 4[2x - 3\ (1 - 3x)]$

Solution:

$5x - 2(3x - 4) > 4[2x - 3\ (1 - 3x)]$
or, $5x - 6x - 8) > 8x - 12 + 36x$
or, $5x - 6x - 8x - 36x > -12 - 8$
or, $-45x > -20$,
Multiplying both side we get –1,
or, $45x < 20$
or, $x < \frac{20}{45}$
i.e. $x < \frac{4}{9}$

Questions for review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Solve the following inequalities:
   (i) $3x - 2 < 4 + 6x$
   (ii) $2x - 3(4 - x) < 7 - 4(1 - 2x)$
   (iii) $2x - 3 + 4(5 - 3x) \geq 4x - 19$
   (iv) $3 - [5x + 11 - 2x(3 + 2)] \leq 11 - 3x[x - 5(3 - x)]$
   (v) $[(5x - 7) / (2x - 3)] - 2(3 - 4x) < 0$

2. Solve each inequality with a sign graph
   (i) $(x + 3)(x + 4)^2 < 0$
   (ii) $(x - 1)(x - 2)(x - 3) < 0$

Lesson-3: Degree of an Equation

After studying this lesson, you should be able to:

- Explain the nature of degree of an equation;
- Solve the simultaneous linear equation.

Nature of Degree of an Equation

An equation involving only one unknown quantity is called ordinary equation. An ordinary equation involving only the first power of the unknown quantity is called 'simple' or 'linear' equation or equation of the first degree. When the highest power of the unknown quantity x is 2, it is called 'quadratic or the second degree equation; when the highest power of the unknown quantity x is 3, the equation is termed as 'cubic' or the third degree equation. When the highest power of x is 4, the equation is called 'biquadratic or the fourth degree equation.

For example,

$2x + 18 = y \rightarrow$ Linear equation

$2x^2 + 5x + 7 = 0 \rightarrow$ Quadratic equation

$x^3 + 5x^2 + 3x + 9 = 0 \rightarrow$ Cubic equation

$x^4 + 10x^3 + 5x^2 + 2x + 10 = 35 \rightarrow$ Biquadratic equation

If an equation in x is unaltered by changing $x$to $\frac{1}{x}$, it is known as a reciprocal equation.

An equation in which the variable occurs as indices or exponents is called an exponential equation.

For example, $3^x = 21$, $81^x = 9^{x+4}$ etc. are called exponential equations.

If more than one unknown quantity are involved, the number in independent equations required for solution is equal to the number of the unknown quantities. Such set of linear equations is called simultaneous linear equations.

The following section of this lesson contains some model applications.

Example-1:

Solve $4x^4 – 16x^3 + 23x^2 – 16x + 4 = 0$

Solution:

$4x^4 – 16x^3 + 23x^2 – 16x + 4 = 0$

Rearranging the terms, we have

$4x^4 – 4 – 16x^3 – 16x + 23x^2 = 0$

Dividing both sides by x^2 we have,

$$4x^2 + \frac{4}{x^2} - 16x - \frac{16}{x} + 23 = 0$$

$$\text{or, } 4\left(x^2 + \frac{1}{x^2}\right) - 16\left(x + \frac{1}{x}\right) + 23 = 0$$

$$\text{or, } 4\left[\left(x + \frac{1}{x}\right)^2 - 2.x.\frac{1}{x}\right] - 16\left(x + \frac{1}{x}\right) + 23 = 0$$

Putting y for $x+\frac{1}{x}$, we have

$4(y^2 – 2) – 16(y) + 23 = 0$

or, $4y^2 – 8 – 16y + 23 = 0$

or, $4y^2 – 16y + 15 = 0$

or, $4y^2 – 10y – 6y + 15 = 0$

or, $2y(2y – 5) – 3(2y – 5) = 0$

or, $(2y – 5)(2y – 3) = 0$

either, $2y - 5 = 0$ or, $2y - 3 = 0$

or, $2y = 5$ or, $2y = 3$

or, $y = \frac{5}{2}$ or, $y = \frac{3}{2}$

When, $y = \frac{5}{2}$, then $x + \frac{1}{x} = \frac{5}{2}$

or, $\frac{x^2+1}{x} = \frac{5}{2}$

or, $2x^2 + 2 = 5x$

or, $2x^2 - 4x - x + 2 = 0$

or, $2x(x-2) - 1(x-2) = 0$

or, $(2x-1)(x-2) = 0$

either, $2x - 1 = 0$ or, $x - 2 = 0$

or, $2x = 1$ $\therefore x = 2$

$\therefore x = \frac{1}{2}$

When, $y = \frac{3}{2}$, then $x + \frac{1}{x} = \frac{3}{2}$

$\frac{x^2+1}{x} = \frac{3}{2}$

or, $2x^2 + 2 = 3x$

or, $2x^2 - 3x + 2 = 0$

We know that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (Here, a = 2, b = –3, c = 2)

$$x = \frac{3 \pm \sqrt{9-16}}{4} = \frac{3 \pm \sqrt{-7}}{4} = \frac{3 \pm \sqrt{7}i}{4} \quad \left(Here,\ i = \sqrt{-1}\right)$$

Hence, $x = 2,\ \frac{1}{2}$ or, $\frac{3 \pm \sqrt{7}i}{4}$

Example-2:

Solve $x + \frac{4}{y} = 1$

$y + \frac{4}{x} = 25$

Solution:

$x + \frac{4}{y} = 1$ .... (i)

$y + \frac{4}{x} = 25$ .... (ii)

From equation (i) $x + \frac{4}{y} = 1$

or, $\frac{xy+4}{y} = 1$

or, $xy + 4 = y$ .... (iii)

From equation (ii) $y + \frac{4}{x} = 25$

$$\text{or, } \frac{xy + 4}{x} = 25$$

or, $xy + 4 = 25x$ .... (iv)

Subtracting the equation (iii) from equation (iv)

$$0 = 25x - y$$

$$\text{or, } y = 25x$$

Putting the value of y in equation (iii), we get

$$x\,(25x) + 4 = 25x$$

$$\text{or, } 25x^2 + 4 = 25x$$

$$\text{or, } 25x^2 - 25x + 4 = 0$$

$$\text{or, } 25x^2 - 20x - 5x + 4 = 0$$

$$\text{or, } 5x\,(5x - 4) - 1\,(5x - 4) = 0$$

$$\text{or, } (5x - 4)\,(5x - 1) = 0$$

either, $5x - 4 = 0$ | or, $5x - 1 = 0$

or, $5x = 4$ | or, $5x = 1$

so, $x = \frac{4}{5}$ | so, $x = \frac{1}{5}$

Now $x = \frac{4}{5}$, then $y = 25 \times \frac{4}{5} = 20$

$x = \frac{1}{5}$, then $y = 25 \times \frac{1}{5} = 5$

Thus, $x = \frac{4}{5}$, $x = \frac{1}{5}$

$y = 20$, $y = 5$

Questions for Review:

These questions are designed to help you assess how far you have understood and apply the learning you have accomplished by answering (in written form) the following questions:

1. Explain the nature of degree of an equation.
2. Solve the following equations:
   (i) $x^4 - 3x^3 - 2x^2 - 3x + 1 = 0$
   (ii) $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{3}{2}$
   $x - y = 3$
   (iii) $3x + 7 - 5z = 0$
   $7x - 3y - 9z = 0$
   $x^2 + 2y^2 + 3z^2 = 23$
3. Solve
   (i) $x^3 + y^3 = 4914$
   $x + y = 18$
   (ii) $27^x = a^y$
   $81^y = 243.3^x$

Lesson–4: Quadratic Equation

After studying this lesson, you should be able to:

- State the nature of quadratic equation;
- Explain the relationship between roots and coefficient of quadratic equation;
- Explain the formation of quadratic equation; and
- Solve the quadratic equation.

Nature of Quadratic Equation

Generally an equation contains the square of unknown variable is called a quadratic equation. The general method of solving a quadratic equation of the form $ax^2 + bx + c = 0$ is given.

Since $ax^2 + bx + c = 0$; by transposition $ax^2 + bx = c$. Hence dividing both sides by a, the co-efficient of x^2 we have,

$$x^2 + \frac{bx}{a} = -\frac{c}{a}$$

Adding to both sides $\left(\frac{b}{2a}\right)^2$

$$x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\text{or, } x^2 + 2.x.\frac{b}{2a} + \left(\frac{b}{2a}\right)^2 = -\frac{b^2}{4a^2} - \frac{c}{a}$$

$$\text{or, } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\text{or, } x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\text{or, } x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\text{or, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus the required roots of $ax^2 + bx + c = 0$ are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Relationship between Roots and the Co-efficient of Quadratic Equation

A quadratic equation has thus exactly two roots. The relationship between the roots and the co-efficient of the quadratic equations as as follows:

Let the quadratic equation is $ax^2 + bx + c = 0,$ then if α and β denote the roots of this quadratic equation, we have

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Therefore by addition,

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-2b}{2a} = -\frac{b}{a}$$

And by multiplication,

$$\alpha\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} \times \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\frac{(-b)^2 - \left(\sqrt{b^2-4ac}\right)^2}{4a^2}$$

$$\frac{b^2 - b^2 + 4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Thus, we have shown that

Sum of the roots $(\alpha+\beta) = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Product of the roots $(\alpha\beta) = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$

Formation of Quadratic Equation

The formation of the quadratic equation whose roots are given can be explained as follows:

Let the general form of quadratic equation is $ax^2 + bx + c = 0$ and αandβ denote the two given roots, where *a, b* and *c* are constants whose values we have to find out.

The sum of the roots $\alpha+\beta = -\frac{b}{a}$,

So, $b = -a(\alpha+\beta)$

And product of the roots, $\alpha\beta = \frac{c}{a}$

so, $c = a\,\alpha\beta$

The above relations imply that

$ax^2 + bx + c = 0$

or, $ax^2+[-a(\alpha+\beta)]x + a\alpha\beta = 0$

or, $ax^2 - a\alpha x - a\beta x + a\alpha\beta = 0$

or, $x^2 - x(\alpha+\beta) + \alpha\beta = 0$ [Dividing both sides by 'a']

So, the required quadratic equation will be

x^2– (sum of the roots)x + (product of the roots) = 0

The following section of this lesson contains some model applications of quadratic equation.

Example–1:

Solve $x^2 - 4x + 13 = 0$

Solution:

$x^2 - 4x + 13 = 0$

We know that $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Here a = 1, b = –4, c = 13

Substituting the given values we have

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4.(1)(13)}}{2.1}$$

$= \frac{4 \pm \sqrt{16-52}}{2}$

$= \frac{4 \pm \sqrt{-36}}{2}$

$= \frac{4 \pm 6i}{2}$ $\left[\text{Since, } i = \sqrt{-1}\right]$

Therefore, x = 2 + 3i or, 2 – 3i

Example–2:

Solve $3x + 2\sqrt{x} = \frac{10\sqrt{x}}{6x\sqrt{x} - 4x}$

Solution:

We have $3x + 2\sqrt{x} = \frac{10\sqrt{x}}{6x\sqrt{x - 4x}}$

$or,\ 3x + 2\sqrt{x} = \frac{2\sqrt{x}.5}{2x\sqrt{(3x - 2\sqrt{x})}}$

$or,\ \left(3x + 2\sqrt{x}\right)\left(3x - 2\sqrt{x}\right) = 5$

$or,\ \left[(3x)^2 - 2\left(2\sqrt{x}\right)^2\right] = 5$

or, $9x^2 - 4x - 5 = 0$

We know that x = $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here a = 9; b = –4; c = –5

Substituting the given values we have

$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(9)(-5)}}{2.9}$

$= \frac{4 \pm \sqrt{16+180}}{18}$

$= \frac{4 \pm \sqrt{196}}{18}$

$= \frac{4 \pm 14}{18}$

$So, x = \frac{18}{18}\ \ or,\ \frac{-10}{18}$

Example–3:

The roots of $x^2 - x + 1 = 0$ are $\alpha$and β; from a quadratic equation whose roots are $\alpha^4 + \beta^4$ and $\alpha^2 + \beta^4$

Solution:

Since α and β are the roots of the equation $x^2 - x + 1 = 0$

Sum of the roots, $\alpha+\beta = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\frac{-1}{1} = 1$

And product of the roots, $\alpha\beta = \dfrac{\text{Constant term}}{\text{Coefficient of } x^2} = \dfrac{1}{1} = 1$

Now, the sum of the roots of the required equation,

$= \alpha^2+\beta^2 + \alpha^2+\beta^4$

$= (\alpha^2+\beta^2) + [(\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2]$

$= [(\alpha+\beta)^2 - 2\alpha\beta] + [(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$

$= (1^2 - 2.1) + [(1)^2 - 2.1]^2 - (1)^2$

$= -1 + 1 - 2 = -3 + 1 = -2.$

And the product of the roots of the required equation.

$= (\alpha^4 + \beta^2)(\alpha^2 + \beta^4)$

$= \alpha^6 + \alpha^4\beta^4 + \alpha^2\beta^2 + \beta^6$

$= (\alpha^6+\beta^6)(\alpha\beta)^4 + (\alpha\beta)^2$

$= (\alpha^2+\beta^2)(\alpha^4 - \alpha^2\beta^2 + \beta^4) + (\alpha\beta)^4 + (\alpha\beta)^2$

$= [(\alpha+\beta)^2 - 2\alpha\beta]\ [(\alpha^2+\beta^2)^2 - 3\alpha^2\beta^2] + (\alpha\beta)^2 + (\alpha\beta)^2$

$= [(\alpha+\beta)^2 - 2\alpha\beta]\ [\{(\alpha+\beta)^2 - 2\alpha\beta\}^2 - 3(\alpha\beta)^2] + (\alpha\beta)^4 + (\alpha\beta)^2$

$= [(1)^2 - 2(1)]\ [\{(1)^2 - 2(1)\}^2 - 3(1)^2] + (1)^2 + (1)^2$

$= (-1)(1-3) + 1 + 1 = 2 + 1 + 1 = 4.$

Hence the required quadratic equation is,

$x^2 -$ (sum of the roots) x + (product of the roots) = 0

or, $x^2 - (-2)x + 4 = 0$

so, $x^2 + 2x + 4 = 0$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. What is a quadratic equation? Give an example
2. State the characteristics of a quadratic equation.
3. If α and β are the roots of $x^2 - px + q = 0$, form a quadratic equation whose roots are $(\alpha\beta - \alpha - \beta)$.
4. If α and β are the roots of $2x^2 + 3x + 7 = 0$, find the values of (1) $\alpha^2+\beta^2$ (ii) α/β (iii) β/α (iv) $\alpha^3+\beta^3$
5. If the equation $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root, then prove that $a + b + c = 0$ and $a = b = c$.
6. Solve: (i) $3x^2 - 2x - 5 = 0$ (ii) $x^2 + 4x + 4 = y$

PERMUTATION AND COMBINATION

5

Unit Highlights

- Lesson – 1: Permutation
- Lesson – 2: Combinations

Technologies Used for Content Delivery

- BOUTUBE
- BOU LMS
- WebTV
- Web Radio
- Mobile Technology with MicroSD Card
- LP+ Office 365
- BTV Program
- Bangladesh Betar Program

Lesson-1: Permutation

After studying this lesson, you should be able to:

- Discuss the nature of permutations;
- Identify some important deduction of permutations;
- Explain the fundamental principles and rules of permutations;
- Highlight on some model application of permutations.

Definition of Permutation

Permutations refer to different arrangements of things from a given lot taken one or more at a time. The number of different arrangements of *r* things taken out of *n* dissimilar things is denoted by ^nP_r.

For example, suppose there are three items *x, y* and *z*.

The different arrangements of these three items taking 2 items at a time are: *xy, yx, yz, zy, zx*and*xz*. Thus $^nP_r = {}^3P_2 = 6$.

Again all the arrangements of these three items taking 3 items at a time are: *xyz, xzy, yzx, yxz, zxy*and*zyx*. Thus $^nP_r = {}^3P_3 = 6$.

Hence it is clear that the number of permutations of 3 things by taking 2 or 3 items at a time is 6.

Fundamental Principles of Permutation

If one operation can be done in *m* different ways where it has been done in any one of these ways, and if a second operation can be done in *n* different ways, then the two operations together can be done in $(m\times n)$ ways.

Permutations of Things All Different

Permutations of *'n'* different things taken *'r'* at a time is denoted by ^nP_r where $r\leq n$. Here, $^nP_r = n.(n-1).(n-2)........(n-r+1)$.

Therefore, the first place can be filled up in *n* ways.
The first two places can be filled up in $n.(n-1)$ ways.
The first three places can be filled up in $n.(n-1).(n-2)$ ways.

Permutation of Things Not All Different

The number of permutation of *'n'* things taken *'r'* at a time in which k_1 elements are of one kind, k_2 elements are of a second kind, k_3 elements are of a third kind and all the rest are different is given by: $^nP_r = \dfrac{r!}{K_1!.K_2!.K_3!......K_n!}$

Circular Permutations

The number of distinct permutations of *n* objects taken *n* at a time on a circle is (n –1)!. In considering the arrangement of keys on a chain or beads on a necklace, two permutations are considered the same if one is obtained from the other by turning the chain or necklace over. In that case there will be $\frac{1}{2}(n-1)!$ways of arranging the objects.

Some Important Deduction of Permutations

(i) $^nP_n = n.(n-1).(n-2)...............$ to *n* factors

$= n.(n-1).(n-2)..........\{n-(n-1)\}$
$= n.(n-1).(n-2)..........1$
$= n.(n-1).(n-2)..........3.2.1.$
$= n!$

(ii) $^{n}P_{n-1} = \frac{n!}{\{n-(n-1)\}!}$ [since, $^{n}P_{r} = \frac{n!}{(n-r)!}$]

$$= \frac{n!}{\{n-n+1)\}!} = \frac{n!}{1!} = n!$$

(iii) $^{n}P_{r} = n.\ ^{n-1}P_{r-1}$

or, $\frac{n!}{(n-r)!} = n.\frac{(n-1)!}{\{(n-1)-(r-1)\}!}$

or, $\frac{n!}{(n-r)!} = n.\frac{(n-1)!}{(n-r)!}$

or, $\frac{n!}{(n-r)!} = \frac{n!}{(n-r)!}$ [since, $n\ (n-1)! = n!$]

$\therefore\ ^{n}P_{r} = n.\ ^{n-1}P_{r-1}$

(iv) $^{n}P_{r} = n.(n-1).(n-2)........ (n-r+1)$

$$= \frac{n(n-1)(n-2)........(n-r+1)(n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

(v) $^{n}P_{r} = {}^{n-1}P_{r} + r.\ ^{n-1}P_{r-1}$

$$= \frac{(n-1)!}{(n-1-r)!} + r.\frac{(n-1)!}{\{(n-1)-(r-1)\}!}$$

$$= \frac{(n-1)!}{(n-1-r)!} + \frac{r.(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-r-1)!} + \frac{r.(n-1)!}{(n-r)(n-r-1)!}$$

$$= \frac{(n-1)!}{(n-r-1)!}\left[1+\frac{r}{(n-r)}\right]$$

$$= \frac{(n-1)!}{(n-r-1)!}\times\left[\frac{n-r+r}{(n-r)}\right]$$

$$= \frac{n(n-1)!}{(n-r)(n-r-1)!}$$

$$= \frac{n!}{(n-r)!} = {}^{n}P_{r}$$

$\therefore\ ^{n-1}P_{r} + r.\ ^{n-1}P_{r-1} = {}^{n}P_{r}$

The following examples contain some model application of permutations.

Example-1:

A store has 8 regular door ways and 5 emergency doors which can be opened only from the inside. In how many ways can a person enter and leave the store?

Solution:

To enter the store, a person may choose any one of 8 different doors. Once inside he may leave by any one of (8+5) =13 doors.

∴ The total number of different ways is (8 × 13) =104.

Example-2:

There are 10 routes for going from a place Chittagong to another place Dhaka and 12 routes for going from Dhaka to a place Khulna. In how many ways can a person go from Chittagong to Khulna Via Dhaka?

Solution:

There are 10 different routes from Chittagong to another place Dhaka, the person can finish the first part of the journey in 10 different ways. And when he has done so in any one way, he will get 12 different ways to finish the second part. Thus one way of going from Chittagong to Dhaka gives rise to 12 different ways of completing the journey from Chittagong to Khulna via Dhaka.

Hence the total number of different ways of finishing both the parts of the journey as desired = (No. of ways for the 1st part × No. of ways for 2nd part) = (10 × 12) = 120.

Example-3:

There are 8 men who are to be appointed as General Manager at 8 branches of a supermarketchain. In how many ways can the 8 men be assigned to the 8 branches?

Solution:

Since every re-arrangement of the 8 men will be considered as a different assignment, the number of ways will be

$$^{8}P_{8} = \frac{8!}{(8-8)!} = (8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 40{,}320 \text{ ways}$$

Example-4:

Six officials of a company are to fly to a conference in Dhaka. Company policy states that no two can fly on the same plane. If there are 9 flights available, how many flight schedules can be established?

Solution:

The number of flight schedule can be established for the six officials in

$$^{9}P_{6} = \frac{9!}{(9-6)!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!} = 60480 \text{ ways}$$

Thus the total number of ways is 60480.

Example-5:

In how many ways can 3 boys and 5 girls be arranged in a row so that all the 3 boys are together?

Solution:

The 3 boys will always be kept together, so we count the 3 boys as one boy. As a result the number of persons involved to be arranged in a row is 6.

They can be arranged in 6! ways = (6 × 5 × 4 × 3 × 2 × 1) = 720 ways.

But these 3 boys themselves can be arranged in 3! ways, i.e. (3 × 2 × 1) = 6 ways.

Hence the required number of arrangement in which the boys are together will be,

= (720 × 6) = 4320 ways.

Example-6:

Out of the letters *P, Q, R, x, y* and *z*, how many arrangements can be made (i) beginning with a capital; (ii) beginning and ending with a capital.

Solution:

(i) One capital letter out of given 3 capital letters can be chosen in $^{3}P_{1}$ = 3 ways. Remaining the other five letters can be arranged among themselves in 5! ways, i.e. in (5 × 4 × 3 × 2 × 1) = 120 ways.

Hence the total number of arrangements beginning with a capital = (120 × 3) = 360.

(ii) Two capital letters out of given 3 capital letters can be chosen in ${}^{3}P_{2} = 6$ ways. For each choice of these two letters, remaining four letters can be arranged in 4! ways, i.e. in (4 × 3 × 2 × 1) = 24 ways.
Therefore the required number of arrangements beginning and ending with a capital
= (6 × 24) = 144.

Example-7:

Six papers are set in an examination of which two are mathematical. In how many different orders can the papers be arranged so that (i) the two mathematical papers are together; (ii) the two mathematical papers are not consecutive.

Solution:

(i) We count the two mathematical papers as one, so that the total number of arrangement can be done in 5! ways, i.e., in (5 × 4 × 3 × 2 × 1) = 120 ways.
Two mathematical papers can be arranged within themselves in 2! = (2 × 1) = 2 ways.
Hence the required number of arrangement in which the mathematical papers are always together is = (120 × 2) = 240.
(ii) Again the total number of possible arrangements is 6! = (6×5×4×3×2×1) = 720 ways.
Hence the total number of arrangements in which mathematical papers are not consecutive is = (720 – 240) = 480 ways.

Example-8:

How many different numbers of 3 digits can be formed from the digits 1, 2, 3, 4, 5 and 6, if digits are not repeated? What will happen if repetitions are allowed?

Solution:

If the repetition of digits is not allowed then the required number of arrangements is, ${}^{6}P_{3}$ = $\frac{6!}{(6-3)!} = \frac{6 \times 5 \times 4 \times 3!}{3!}$ = 120 ways.

If the repetition of digits are allowed then the required number of arrangements is,
= $(n \times n \times n) = (6 \times 6 \times 6) = 216$ ways.
Therefore 120 and 216 different numbers can be formed respectively by repeating and not repeating digits 1, 2, 3, 4, 5 and 6.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Indicate how many 4 digit numbers smaller than 6,000 can be formed from the digits 2, 4, 5, 6, 8, 9?
2. Find the number of arrangements than can be made out of the letters of the word "ASSASSINATION".
3. Indicate how many 5 digit numbers can be formed from the digits 2, 3, 5, 6, 8, 9 where 6 and 9 must be included in all cases.
4. In how many ways can 6 persons formed a ring?
5. How many different arrangements can be made of all the letters of the word "ACCOUNTANTS"? In how many of them the vowels stand together?
6. In how many ways 3 boys and 5 girls be arranged in a row so that all the 5 girls, are together?
7. In how many ways can the letters of the word "EQUATION" be arranged so that the consonants may occupy only odd positions?
8. In how many ways can seven supervisors and six engineers sit for a round table discussion so that no two supervisors are setting together?
9. Find the number of permutations of the word ENGINEERING.

Lesson-2: Combinations

After studying this lesson, you should be able to:

- State the nature of combinations;
- Explain the important deductions of combinations;
- Highlight on some model applications of combinations.

Definition of Combination

Combination refers to different set of groups made out of a given lot, without repeating an element, taking one or more of them at a time. In other words, each of the groups which can be formed out of *n* things taking *r* at a time without regarding the order of things in each group is termed as combination. It is denoted by ^nC_r.

For example, suppose there are three things *x, y* and *z*.

The combinations of 3 things taken 2 things at a time are: *xy, yz, zx*

Thus $^nC_r = {}^3C_2 = 3$.

Some Important Deductions of Combinations

(i) $^nC_r = \dfrac{n!}{r!(n-r)!}$

Generally ^nC_r combinations would produce ($^nC_r \times r!$) permutations; i.e., ($^nC_r \times r!$) = ^nP_r .

Hence, ($^nC_r \times r!$) = ^nP_r

$$^nC_r = \frac{^nP_r}{r!} = \frac{n.(n-1).(n-2)..................(n-r+1)}{r!}$$

$$^nC_r = \frac{n.(n-1).(n-2)..................(n-r+1).(n-r)!}{r.(n-r)!}$$

$$^nC_r = \frac{n!}{r!(n-r)!}$$

(ii) $^nC_0 = \dfrac{n!}{0!(n-0)!} = \dfrac{n!}{n!} = 1$ [since 0! = 1]

(iii) $^nC_1 = \dfrac{n!}{1!(n-1)!} = \dfrac{n.(n-1)!}{1!(n-1)!} = n$

(iv) $^nC_n = \dfrac{n!}{n!(n-n)!} = \dfrac{n!}{n!} = 1$

(v) $^nC_{n-1} = \dfrac{n!}{(n-1)!\{n-(n-1)\}!} = \dfrac{n.(n-1)!}{(n-1)!(n-n+1)!} = n$

$\therefore {}^nC_1 = {}^nC_{n-1}$

(vi) $^nC_r = {}^nC_{n-r}$

$$= \frac{n!}{(n-r)!\{n-(n-r)\}!} = \frac{n!}{(n-r)!r!} = {}^nC_r$$

Therefore, $^nC_r = {}^nC_{n-r}$

(vii) Prove that ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$

We know that ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

$$\therefore {}^nC_r + {}^nC_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!\{(n-(r-1)\}!}$$

$$= \frac{n!}{r.(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{(n-r+1)}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n-r+1+r}{r.(n-r+1)}\right]$$

$$= \frac{(n+1).n!}{r.(r-1)!.(n-r+1).(n-r)!}$$

$$= \frac{(n+1)!}{r!.(n-r+1)!}$$

$$= \frac{(n+1)!}{r!.\{(n+1)-r\}!} = {}^{n+1}C_r \text{ (Proved).}$$

The following examples illustrate some model applications of combinations.

Example-1:

Find out the number of ways in which a cricket team consisting of 11 players can be selected from 14 players. Also find out how many of these ways (i) will include captain (ii) will not include captain?

Solution:

The numbers of ways in which 11 out of 14 players can be selected are

$${}^nC_r = {}^{14}C_{11} = \frac{14!}{11!(14-11)!} = \frac{14\times 13\times 12\times 11!}{11!\times 3\times 2\times 1} = 364$$

(i) As captain is to be kept in every combination, we are to choose 10 out of the remaining 13 players. Therefore the required number of ways,

$${}^{13}C_{10} = \frac{13!}{10!(13-10)!} = \frac{13\times 12\times 11\times 10!}{10!\times 3\times 2\times 1} = 286 \text{ ways}$$

(ii) In this case as captain is to be excluded, therefore, we are to choose 11 out of remaining 13 players which can be done in,

$${}^{13}C_{11} = \frac{13!}{11!(13-11)!} = \frac{13\times 12\times 11!}{11!\times 2\times 1} = 78 \text{ ways.}$$

Example-2:

Out of 17 consonants and 5 vowels, how many different words can be formed each containing 3 consonants and 2 vowels?

Solution:

3 consonants can be selected out of 17 in ${}^{17}C_3$ ways and 2 vowels can be selected out of 5 in ${}^5C_2$ ways.

$\therefore$ The number of selections having 3 consonants and 2 vowels = ${}^{17}C_3 \times {}^{5}C_2$ ways.
Each of these selections contains 5 letters which can be arranged among themselves in 5! ways.
Therefore the total number of words = ${}^{17}C_3 \times {}^{5}C_2 \times 5!$

$$= \frac{17!}{3!(17-3)!} \times \frac{5!}{2!(5-2)!} \times 5!$$

$$= \frac{17 \times 16 \times 15 \times 14!}{3 \times 2 \times 1 \times 14!} \times \frac{5 \times 4 \times 3!}{2 \times 1 \times 3!} \times 5.4.3.2.1 = 8{,}16{,}000.$$

Example-3:
From 6 boys and 4 girls, a committee of 6 is to be formed. In how many ways can this be done if the committee contains (i) exactly 2 girls, or (ii) at least 2 girls?

Solution:
(i) The committee of 6 is to contain 2 girls and 4 boys.

Therefore 2 girls can be selected out of 4 girls in ${}^{4}C_2 = \frac{4.3.2!}{2.1.2!} = 6$ ways.

The remaining 4 boys can be selected out of 6 in ${}^{6}C_4 = \frac{6.5.4!}{4!.2.1} = 15$ ways

Therefore the required number of ways = (6 × 15) = 90 ways.
(ii) In this case the committee of 6 can be formed in the following ways.
(a) 2 girls and 4 boys, (b) 3 girls and 3 boys and (c) 4 girls and 2 boys.
We now consider all these 3 cases:
In case of (a) the committee of 6 can be formed as explained above in ${}^{4}C_2 \times {}^{6}C_4$ ways.
Accordingly there are ${}^{4}C_3 \times {}^{6}C_3$ and ${}^{4}C_4 \times {}^{6}C_2$ ways of forming the committee in cases of (b) and (c) respectively.
Hence, the total number of different ways

$$= ({}^{4}C_2 \times {}^{6}C_4) + ({}^{4}C_3 \times {}^{6}C_3) + ({}^{4}C_4 \times {}^{6}C_2)$$

$$= \left(\frac{4!}{2!(4-2)!} \times \frac{6!}{4!(6-4)!}\right) + \left(\frac{4!}{3!(4-3)!} \times \frac{6!}{3!(6-3)!}\right) + \left(\frac{4!}{4!(4-4)!} \times \frac{6!}{2!(6-2)!}\right)$$

$$= [(6 \times 15) + (4 \times 20) + (1 \times 15)] = (90 + 80 + 15) = 185.$$

Example-4:
A cricket team consisting of 11 players is to be formed from 16 players of whom 4 can be bowlers and 2 can keep wicket and the rest can neither be bowler nor keep wicket. In how many different ways can a team be formed so that the teams contain (i) exactly 3 bowlers and 1 wicket keeper, (ii) at least 3 bowlers and at least 1 wicket keeper?

Solution:
(i) A cricket team of 11 is to be formed with exactly 3 bowlers and 1 wicket keeper.
3 bowlers can be selected out of 4 in ${}^{4}C_3$ ways, 1 wicket keeper can be selected out of 2 in ${}^{2}C_1$ ways and the other 7 players can be selected from the remaining 10 players in ${}^{10}C_7$ ways.
Hence the total number of ways in which the cricket team can be formed

$$= {}^{4}C_3 \times {}^{2}C_1 \times {}^{10}C_7$$

$$= \frac{4!}{3!(4-3)!} \times \frac{2!}{1!(2-1)!} \times \frac{10!}{7!(10-7)!}$$

$$= (4 \times 2 \times 120) = 960.$$

(ii) Since at least 3 bowlers and at least one wicket keeper is to be included in the cricket team of 11 players, the team can be formed by choosing.

(a) 3 bowlers, 1 wicket keeper and 7 other players.
(b) 3 bowlers, 2 wicket keeper and 6 other players.
(c) 4 bowlers, 1 wicket keeper and 6 other players.
(d) 4 bowlers, 2 wicket keeper and 5 other players.

We now consider all these 4 cases.

(a) 3 bowlers, 1 wicket keeper and 7 other players can be selected in

$$= {}^4C_3 \times {}^2C_1 \times {}^{10}C_7 = \frac{4!}{3!(4-3)!} \times \frac{2!}{1!(2-1)!} \times \frac{10!}{7!(10-7)!}$$

= (4 × 2 × 120) = 960 ways.

(b) 3 bowlers, 2 wicket keeper and 6 other players can be selected in

$= {}^4C_3 \times {}^2C_2 \times {}^{10}C_6 = (4 \times 1 \times 210) = 840$ ways.

(c) 4 bowlers, 1 wicket keeper and 6 other players can be selected in

$= {}^4C_4 \times {}^2C_1 \times {}^{10}C_6 = (1 \times 2 \times 210) = 420$ ways

(d) 4 bowlers, 2 wicket keeper and 5 other players can be selected in

$= {}^4C_4 \times {}^2C_2 \times {}^{10}C_5 = (1 \times 1 \times 252) = 252$ ways

Therefore, the total number of ways

= (960 + 840 + 420 + 252) = 2472.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. A committee of 5 is to be formed from 14 students. How many different ways this can be done so as always to (i) include 2 particular students; and (ii) exclude 3 particular students?
2. A question paper contains six questions, each having an alternative. In how many ways can an examinee answer one or more questions?
3. A committee consists of 5 members is to be formed out of 6 men and 4 women. How many types of committees can be formed so that at least 2 women are always there?
4. Out of 10 consonants and 4 vowels, how many words can be formed each containing 3 consonants and 2 vowels?
5. From 6 boys and 4 girls, 5 are to be selected for admission into a particular course. In how many ways can this be done if there must be exactly 2 girls?
6. In how many ways a committee of 5 members can be formed out of 8 professors? How often will each professor be selected? If one particular professor is always included, what will be the number of ways? In how many ways the committee can be formed if one particular professor is always excluded?

MATHEMATICS INDUCTION, SEQUENCE, AND SERIES

6

Unit Highlights

- Lessons – 1 & 2: Mathematics Induction
- Lessons – 3 & 4: Sequence and Series

Technologies Used for Content Delivery

- BOUTUBE
- BOU LMS
- WebTV
- Web Radio
- Mobile Technology with MicroSD Card
- LP+ Office 365
- BTV Program
- Bangladesh Betar Program

Lessons-1 and 2: Mathematics Induction

After studying of this lesson, you should be able to:

- Define the mathematics induction;
- Explain the principles of mathematics induction;
- Solve the problems of mathematics induction.

Link of this chapter.

https://byjus.com/maths/principle-of-mathematical-induction-learn-examples/

Mathematical Induction is a technique of proving a statement, theorem or formula which is thought to be true, for each and every natural number n. By generalizing this in form of a principle which we would use to prove any mathematical statement is '**Principle of Mathematical Induction**'.

For example: $1^3 + 2^3 + 3^3 + ….. + n^3 = (n(n+1) / 2)^2$, the statement is considered here as true for all the values of natural numbers.

Mathematical Induction is a powerful and elegant technique for proving certain types of mathematical statements: general propositions which assert that something is true for all positive integers or for all positive integers from some point on. Let us look at some examples of the type of result that can be proved by induction.

Proposition 1. The sum of the first n positive integers (1, 2, 3,...) is 1 2n(n + 1).

Proposition 2. In a convex polygon with n vertices, the greatest number of diagonal that can be drawn is 1 2n(n − 3).

Principle of Mathematical Induction Solution and Proof

Consider a statement P(n), where n is a natural number. Then to determine the validity of P(n) for every n, use the following principle:

Step 1: Check whether the given statement is true for n = 1.

Step 2: Assume that given statement P(n) is also true for n = k, where k is any positive integer.

Step 3: Prove that the result is true for P(k+1) for any positive integer k.

If the above-mentioned conditions are satisfied, then it can be concluded that P(n) is true for all n natural numbers.

Proof:

The first step of the principle is a *factual statement* and the second step is a *conditional one*. According to this if the given statement is true for some positive integer k only then it can be concluded that the statement P(n) is valid for n = k + 1.

This is also known as the *inductive step* and the assumption that P(n) is true for n=k is known as the *inductive hypothesis*.

Solved problems

Example 1: Prove that the sum of cubes of n natural numbers is equal to $([n(n+1)]/2)^2$ for all n natural numbers.

Solution:

In the given statement we are asked to prove:

$1^3+2^3+3^3+\cdots+n^3 = ([n(n+1)]/2)^2$

Step 1: Now with the help of the principle of induction in Maths, let us check the validity of the given statement P(n) for n=1.

$P(1)=([1(1+1)]/2)^2 = (2/2)^2 = 1^2 = 1$.

This is true.

Step 2: Now as the given statement is true for n=1, we shall move forward and try proving this for n=k, i.e.,

$1^3+2^3+3^3+\cdots+k^3= ([k(k+1)]/2)^2$.

Step 3: Let us now try to establish that P(k+1) is also true.

13+23+33+….k3+(k+1)3=[k(k+1)2]2+(k+1)(k+1)2

=(k+1)2[k24+(k+1)]   =(k+1)2(k2+4k+44)

=(k+1)2(k+22)2   =[(k+1)(k+2)2]2

Example 2: **Show that $1 + 3 + 5 + \ldots + (2n-1) = n^2$**

Solution:

Step 1: Result is true for n = 1
That is $1 = (1)^2$ (True)

Step 2: Assume that result is true for n = k
$1 + 3 + 5 + \ldots + (2k-1) = k^2$

Step 3: Check for n = k + 1

i.e. $1 + 3 + 5 + \ldots + (2(k+1)-1) = (k+1)^2$

We can write the above equation as,

$1 + 3 + 5 + \ldots + (2k-1) + (2(k+1)-1) = (k+1)^2$

Using step 2 result, we get

$k^2 + (2(k+1)-1) = (k+1)^2$

$k^2 + 2k + 2 -1 = (k+1)^2$

$k^2 + 2k + 1 = (k+1)^2$

$(k+1)^2 = (k+1)^2$

L.H.S. and R.H.S. are same.

So the result is true for n = k+1

By mathematical induction, the statement is true.

We see that the given statement is also true for n=k+1. Hence we can say that by the principle of mathematical induction this statement is valid for all natural numbers n.

Example 3:

Show that $2^{2n}-1$ is divisible by 3 using the principles of mathematical induction.
To prove: $2^{2n}-1$ is divisible by 3

Assume that the given statement be P(k)

Thus, the statement can be written as $P(k) = 2^{2n}-1$ is divisible by 3, for every natural number

Step 1: In step 1, assume n= 1, so that the given statement can be written as

$P(1) = 2^{2(1)}-1 = 4-1 = 3$. So 3 is divisible by 3. (i.e.3/3 = 1)

Step 2: Now, assume that P(n) is true for all the natural numbers, say k

Hence, the given statement can be written as

$P(k) = 2^{2k}-1$ is divisible by 3.

It means that $2^{2k}-1 = 3a$ (where a belongs to natural number)

Now, we need to prove the statement is true for n= k+1

Hence,

$P(k+1) = 2^{2(k+1)}-1$

$P(k+1)= 2^{2k+2}-1$

$P(k+1)= 2^{2k}. 2^2 – 1$

$P(k+1)= (2^{2k}. 4)-1$

$P(k+1) =3.^{2k}+(2^{2k}-1)$

The above expression can be written as

$P(k+1) =3.2^{2k}+ 3a$

Now, take 3 outside, we get

$P(k+1) = 3(2^{2k}+ a)= 3b$, where "b" belongs to natural number

It is proved that p(k+1) holds true, whenever the statement P(k) is true.

Thus, $2^{2n}-1$ is divisible by 3 is proved using the principles of mathematical induction

Questions for Review

1. Prove that 1 × 1! + 2 × 2! + 3 × 3! + … + n × n! = (n + 1)! – 1 for all natural numbers using the principles of mathematical induction.
2. Prove that $4^n – 1$ is divisible by 3 using the principle of mathematical induction

Use the principles of mathematical induction to show that 2 + 4 + 6 + … + 2n = $n^2 + n$, for all natural numbers

Lessons-3 and 4: Sequence and Series

After studying of this lesson, you should be able to:

- Define the sequence and series;
- Explain the types of sequence and series
- Know the formula of sequence and series ;
- Distinguish between sequence and series;
- Solve the problems of sequence and series.

Link of this chapter.
https://byjus.com/maths/sequence-and-series/

Sequence

A succession of numbers arranged in a definite order according to a given certain rule is called sequence. A sequence is either finite or infinite depending upon the number of terms in a sequence.

Sequences The list of positive odd numbers 1, 3, 5, 7, 9, ... is an example of a typical infinite sequence. The dots indicate that the sequence continues forever, with no last term. We will use the symbol an to denote the nth term of a given sequence. Thus, in this example, a1 = 1, a2 = 3, a3 = 5 and so on; the first term is a1 = 1, but there is no last term. The list of positive odd numbers less than 100 1, 3, 5, 7, ..., 99 is an example of a typical finite sequence. The first term of this sequence is 1 and the last term is 99. This sequence contains 50 terms.

There are several ways to display a sequence:

• write out the first few terms

• give a formula for the general term

• give a recurrence relation.

Writing out the first few terms is not a good method, since you have to 'believe' there is some clearly defined pattern, and there may be many such patterns present.

For example, if we simply write

1, 2, 4, ...

then the next term might be 8 (powers of two), or possibly 7 (Lazy Caterer's sequence), or perhaps even 23 if there is some more complicated pattern going on. Hence, if the first few terms only are given, some rule should also be given as to how to uniquely determine the next term in the sequence.

A much better way to describe a sequence is to give a formula for the nth term an. This is also called a formula for the general term. For example,

an = 2n −1

is a formula for the general term in the sequence of odd numbers 1, 3, 5,.... From the formula, we can, for example, write down the 10th term, since a10 = 2×10−1 = 19.

In some cases it is not easy, or even possible, to give an explicit formula for an. In such cases, it may be possible to determine a particular term in the sequence in terms of some of the preceding terms. This relationship is often referred to as a recurrence.

For example, the sequence of positive odd numbers may be defined by

a1 = 1 and an+1 = an +2, for n ≥ 1.

The initial term is a1 = 1, and the recurrence tells us that we need to add two to each term to obtain the next term. The Fibonacci sequence comprises the numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

where each term is the sum of the two preceding terms. This can be described by setting

a1 = a2 = 1 and

an+2 = an+1 + an, for n ≥ 1.

https://byjus.com/maths/sequence-and-series/

Sequence and series are the basic topics in Arithmetic. An itemized collection of elements in which repetitions of any sort are allowed is known as a sequence, whereas a series is the sum of all elements. An arithmetic progression is one of the common examples of sequence and series.

- In short, a **sequence** is a list of items/objects which have been arranged in a sequential way.
- A **series** can be highly generalized as the sum of all the terms in a sequence. However, there has to be a definite relationship between all the terms of the sequence.

The fundamentals could be better understood by solving problems based on the formulas. They are very similar to sets but the primary difference is that in a sequence, individual terms can occur repeatedly in various positions. The length of a sequence is equal to the number of terms and it can be either finite or infinite. This concept is explained in a detailed manner in Class 11 Maths. With the help of definition, formulas and examples we are going to discuss here the concepts of sequence as well as series.

Sequence and Series Definition

A sequence is an arrangement of any objects or a set of numbers in a particular order followed by some rule. If a_1, a_2, a_3, a_4,……… etc. denote the terms of a sequence, then 1,2,3,4,…..denotes the position of the term.

A sequence can be defined based on the number of terms i.e. either finite sequence or infinite sequence.

If a_1, a_2, a_3, a_4, ……. is a sequence, then the corresponding series is given by

$S_N = a_1+a_2+a_3 + .. + a_N$

Note: The series is finite or infinite depending if the sequence is finite or infinite.

Types of Sequence and Series

Some of the most common examples of sequences are:

- Arithmetic Sequences
- Geometric Sequences
- Harmonic Sequences
- Fibonacci Numbers

Arithmetic Sequences

A sequence in which every term is created by adding or subtracting a definite number to the preceding number is an arithmetic sequence.

Geometric Sequences

A sequence in which every term is obtained by multiplying or dividing a definite number with the preceding number is known as a geometric sequence.

Harmonic Sequences

A series of numbers is said to be in harmonic sequence if the reciprocals of all the elements of the sequence form an arithmetic sequence.

Fibonacci Numbers

Fibonacci numbers form an interesting sequence of numbers in which each element is obtained by adding two preceding elements and the sequence starts with 0 and 1. Sequence is defined as, F_0 = 0 and F_1 = 1 and $F_n = F_{n-1} + F_{n-2}$

Sequence and Series Formulas

List of some basic formula of arithmetic progression and geometric progression are

	Arithmetic Progression	**Geometric Progression**
Sequence	a, a+d, a+2d,……,a+(n-1)d,….	a, ar, ar^2,….,$ar^{(n-1)}$,…
Common Difference or Ratio	Successive term – Preceding term Common difference = d = $a_2 - a_1$	Successive term/Preceding term Common ratio = r = $ar^{(n-1)}/ar^{(n-2)}$
General Term (nth Term)	$a_n = a + (n-1)d$	$a_n = ar^{(n-1)}$
nth term from the last term	$a_n = l - (n-1)d$	$a_n = l/r^{(n-1)}$
Sum of first n terms	$s_n = n/2(2a + (n-1)d)$	$s_n = a(1 - r^n)/(1 - r)$ if $\|r\| < 1$ $s_n = a(r^n - 1)/(r - 1)$ if $\|r\| > 1$

*Here, a = first term, d = common difference, r = common ratio, n = position of term, l = last term

Difference Between Sequences and Series

Let us find out how a sequence can be differentiated with series.

Sequences	**Series**
Set of elements that follow a pattern	Sum of elements of the sequence
Order of elements is important	Order of elements is not so important
Finite sequence: 1,2,3,4,5	Finite series: 1+2+3+4+5
Infinite sequence: 1,2,3,4,……	Infinite Series: 1+2+3+4+……

Sequence and Series Examples

Example 1: **If 4,7,10,13,16,19,22……is a sequence, Find:**

1. **Common difference**
2. **nth term**
3. **21st term**

Solution: Given sequence is, 4,7,10,13,16,19,22……

a) The common difference = **7 – 4 = 3**

b) The nth term of the arithmetic sequence is denoted by the term T_n and is given by $T_n = a + (n-1)d$, where "a" is the first term and d is the common difference. $T_n = 4 + (n - 1)3 = 4 + 3n - 3 = 3n + 1$

c) 21st term as: $\mathbf{T_{21} = 4 + (21-1)3 = 4+60 = 64}$.

Example 2: Consider the sequence 1, 4, 16, 64, 256, 1024….. Find the common ratio and 9th term.

Solution: The common ratio (r) = 4/1 = 4

The preceding term is multiplied by 4 to obtain the next term.

The nth term of the geometric sequence is denoted by the term T_n and is given by $T_n = ar^{(n-1)}$ where a is the first term and r is the common ratio.

Here a = 1, r = 4 and n = 9

So, 9th term is can be calculated as $T_9 = 1 * (4)^{(9-1)} = 4^8 = 65536$.

- **Example 3:** What will be the 15^{th} term of the arithmetic sequence -3, -(1/2), 2…. using sequence and series formula?

  Solution: Given a = -3, d = -(1/2) -(-3) = 5/2, n = 15

  Using the formula for n^{th} term of an arithmetic sequence: $a_n = a+(n-1)d$

  Putting the known values:

  $a_{15} = -3 +(15-1)\ 5/2$

  $a_{15} = 32$

  Answer: The 15^{th} term of the given arithmetic sequence is 32.

- **Example 4:** Find the next term of the given geometric sequence: 1, 1/2, 1/4, 1/8 ... using sequence and series formula

  Solution:

  Given: a = 1, r = (1/2)/1 = 1/2

  To find: 5^{th} term

  Using the formula for the n^{th} term of a geometric sequence and series:

  $a_n = ar^{(n-1)}$

  Putting the known values in the formula:

  $a_5 = 1(1/2)^{(5-1)}$

  $a_5 = (1/2)^{(4)}$

  $a_5 = 1/16$

  Answer: The next term of the sequence is 1/16.

- **Example 5:** Find the sum of the infinite geometric series -1 + 1/2 - 1/4 + 1/8 - 1/16 + ...

  Solution:

  The common ratio of the given series is, r = -1/2.

  Here, |r| = |-1/2| = 1/2 < 1.

  Using the sequence and series formulas,

  Sum of the given series = a / (1 - r)

  = -1 / (1 - (-1/2))

  = -1 / (3/2)

  = -2/3

  Answer: -2/3

Questions for review:

1. Insert two geometric means between 9 and 243.
2. If p, q, r are in AP then prove that pth, qth and rth term of any GP are in GP.
3. The first term of a GP is 27 and its 8th term is 1/81. Find the sum of the first 10 terms of the GP.
4. Find the sum of the infinite geometric series (√2 + 1) + 1 + (√2 – 1) + …
5. Find the sum of integers from 1 to 100 divisible by 2 or 5.
6. Find the first term of the arithmetic sequence if the common difference is −4−4 and the seventh term is 66.

ARITHMETIC AND GEOMETRIC PROGRESSION

7

Unit Highlights

- Lessons – 1 & 2: Arithmetic Progression and Geometric Progression

Technologies Used for Content Delivery

- BOUTUBE
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- LP+ Office 365
- BTV Program
- Bangladesh Betar Program

Lessons-1 and 2: Arithmetic Progression and Geometric Progression

After studying of this lesson, you should be able to:

- Define arithmetic and geometric progression;
- Define sequence and series;
- Distinguish between arithmetic and geometric progression;
- Solve the problems of AP and Gp progression.

Link of this chapter.
https://www.geeksforgeeks.org/arithmetic-progression-and-geometric-progression/

Arithmetic Progression and Geometric Progression: The word “sequence” in English means a collection of some numbers or objects in such a way that it has a first member, a second member, and so on. Sequences can be of anything, for example. – January, February, ….is the sequence of months in a year. Sequences come into use in actual real lives of people every day. Days of a week can also be considered as a sequence. Thus, it becomes essential to study the sequences and find patterns in them to predict the sequence’s next terms and extract information from them.

Sequences

Let’s consider a sequence: 2,4,6,8 and so on. The various numbers occurring in it are called its terms. They are denoted by a1, a2, a3 … an. The subscripts denote the nth term. The nth term of the sequence is also called the general term of the sequence because we can derive every other term from it by putting different values of n. Here in this case,

- a1 = 2, a2 = 4, a3 = 6 and so on…

A sequence with a finite number of terms is called a finite sequence and similarly, a sequence with an infinite number of terms is called an infinite sequence.
A sequence can be regarded as a function whose domain is the set of natural numbers or some subset of it. Sometimes, we use the functional notation a(n) for an.

Series

For a given sequence a1, a2, a3 … an. The expression given below is called a series. A series can be infinite or finite depending upon the number of terms its sequence has. ∑ is the common notation used to denote the series. This indicates the summation involved.

- Σi=1nai Σ*i*=1*nai* = a1 + a2+ a3 +… an

These concepts give rise to the sequences known as arithmetic progression and geometric progression.

What is Arithmetic Progression (A.P)

Consider a sequence 1, 3, 5, 7, ….. Notice that in this sequence, the difference between successive terms is constant. This means that at each step a constant value is being added to each term of this sequence. A sequence a1, a2, a3 … an can be called an arithmetic progression if an+1 = an + d where n is any natural number. In such a series, a1 is called the first term, and the constant term d is called the common difference of A.P. So, an AP looks like,

- a, a + d, a + 2d, a + 3d ….. and so on.
  The nth for AP can be defined as,
- an = a1 + (n-1)d
- Sum of n terms of an AP is given by,
- Sn = n2[a+(n−1)d]2*n*[*a*+(*n*−1)*d*]
- or
- Sn = n2[a+l]2*n*[*a*+*l*]

What is Geometric Progression (G.P)

Consider the following sequence, 2, 4, 8, 16 ….. It is clear here, that each term is being multiplied by 2 in this sequence. Such sequences where successive terms are multiplied by a constant number are called geometric progressions. In a more general way, a sequence a1, a2, a3 … an can be called a geometric progression if an+1 = an. r where n is any natural number. In such a series, a1 is called the first term, and the constant term r is called the common ratio of G.P. So, a GP looks like,

- a, ar, ar2, arn….. and so on.

The nth for GP can be defined as,

- an = a1rn-1

In general, GP can be finite and infinite but in the case of infinite GP, the common ratio must be between 0 and 1, or else the values of GP go up to infinity. Sum of GP consists of two cases:

Let's denote the Sn are a + ar + ar2 + ….. arn

Case 1: If r = 1, the series collapses to

- a, a, a, a … and so on.
- Sn = na

Case 2: If r≠1, the series stays the same,

- a + ar + ar2 + ….. arn
- Sn = a(1–rn)1–r1–*ra*(1–*rn*)

Let's look at some word problems related to these concepts

Arithmetic Progression vs Geometric Progression

Aspect	Arithmetic Progression (AP)	Geometric Progression (GP)
Definition	Sequence where the difference between consecutive terms is constant.	Sequence where the ratio between consecutive terms is constant.
Common Term	Common Difference (*d*)	Common Ratio (r)
General Formula	an = a + (n-1)d	an = a . rn-1
Sum of First n Terms	Sn = n/2 (2a + (n-1)d)	Sn = a rn – 1/r -1
Identification	Difference between consecutive terms is constant.	Ratio between consecutive terms is constant.
Graphical Representation	Linear sequence	Exponential sequence
Examples	3, 7, 11, 15, 19 (Common difference d = 4)	2, 6, 18, 54 (Common ratio *r* = 3)
Applications	Used in problems involving linear relationships, such as scheduling and financial calculations.	Used in problems involving exponential growth or decay, such as population growth and interest calculations.
Nature of Sequence	Additive	Multiplicative

Solved Examples on AP and GP

Example 1: A bitcoin stock started at $5. After that, every day it rises by $2. Find the stock price at the end of 16th day.

Answer:

In the above question, each time a constant number is added to the previous term to make a new term. This is an AP.

5, 7, 9, … and so on.

Using the formula for nth term of AP.

$an = a1 + (n-1)d$

Here a1 denotes the first term and d denotes the common difference. In this case ,

$a1 = 5, d = 2$ *and* $n = 16$

$a10 = a1 + (16-1)d$

$\Rightarrow a10 = 5 + (15)2$

$\Rightarrow a10 = 5 + 30$

$\Rightarrow a10 = 35$

Thus, the stock prices are at $35.

Example 2: A person planted 3 trees at his son's birth. After that, on subsequent birthdays he planted 5 more trees every year. Find the number of trees in his backyard when his son is 10 years old.

Answer:

In the above question, each time a constant number is added to the previous term to make a new term. This is an AP.

3, 8, 13, … and so on.

Using the formula for nth term of AP.

$an = a1 + (n-1)d$

Here a1 denotes the first term and d denotes the common difference. In this case,

$a1 = 3, d = 5$ *and* $n = 10$

$a10 = a1 + (10-1)d$

$\Rightarrow a10 = a1 + (9)d$

$\Rightarrow a10 = 3 + 9(5)$

$\Rightarrow a10 = 3 + 45$

$\Rightarrow a10 = 48$

Thus, there are 48 trees in his backyard now.

Example 3: English rock band the1975 released a new album in summer, and they opened with 100,000 copies sold within one day. Now the album is topping charts and everyday they sell 20,000 copies more than the previous day. Find the total album sales in a week.

Answer:

In the above question, each time a constant number is added to the previous term to make a new term. This is an AP.

100,000; 120,000; 140,000; … and so on.

Goal is to calculate the sum of the sequence at the end of 10th day.

Using the formula for sum till nth term of AP.

Sn = n2[a+(n−1)d]2n[a+(n−1)d]

Here a denotes the first term and d denotes the common difference. In this case,

a = 100,000, d = 20,000 and n = 7

Sn = n2[a+(n−1)d]2n[a+(n−1)d]

⇒S7 = 72[100000+(7−1)(20000)]27[100000+(7−1)(20000)]

⇒S7 = 72[100000+(6)(20000)]27[100000+(6)(20000)]

⇒S7 = 72[100000+(120000)]27[100000+(120000)]

⇒S7 = 72[220000]27[220000]

⇒S7 = 7(110000)7(110000)

⇒S7 = 770000

Thus, the total album sale is 770,000.

Example 4: The deer population is increasing in Corbett National Park. In year 2015 it was 1000, since then it has been increasing, and it becomes 2 times every year. Find the population in 2021.

Solution.

Here, every year the population becomes 2 times. A constant number is being multiplied to the previous term to get the new term. This is a geometric progression.

1000, 2000 … and so on.

Here a = 1000 and r= 2

Using the formula for nth term of the GP

an = a1rn-1

In 2021, n = 7. Plugging in the values in the formula

an = a1rn-1

⇒an = (1000)(2)(7-1)

⇒an = (1000)(2)6

⇒an = (1000)(64)

⇒an = 64000

There must be 64,000 deer in Corbett National Park now.

Question for review:

Practice Problems on AP and GP

1. Find the 10th term of the arithmetic progression (AP) where the first term a = 3 and the common difference d = 5.
2. Find the sum of the first 20 terms of the arithmetic progression where the first term a = 2 and the common difference d = 3.
3. Given the arithmetic progression 7, 13, 19,…, find the common difference.
4. Find the 5th term of the geometric progression (GP) where the first term a = 2 and the common ratio r = 3.

FUNCTIONS, LIMIT, AND CONTINUITY OF A FUNCTION

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Unit Highlights

- Lesson – 1: Functions
- Lesson – 2: Limit
- Lesson – 3: Continuity

Technologies Used for Content Delivery

- BOUTUBE
- BOU LMS
- WebTV
- Web Radio
- Mobile Technology with MicroSD Card
- LP+ Office 365
- BTV Program
- Bangladesh Betar Program

Lesson-1: Functions

After studying this lesson, you should be able to:

- Discuss the nature of variable and constants;
- State the functions and its classification;
- Highlights on some worked out examples related to the functions.

Introduction:
First of all we have to know some important terms, which are frequently used in this lesson. These are:

- **Constant**
  A constant is a symbol - which never changes over the sets of mathematical operation. For example, 1, 2, 3 are constants. The letter *a, b, c* --- are also considered as constants which are specially know as arbitrary constants.
- **Variables**
  A symbol capable of assuming different values is called a variable. Variable are usually denoted by the letters of the alphabet; i.e., *x, y, z.*
- **Independent Variable**
  A variable to which any value can be assigned is called an independent variable. Independent variables are the causes and the dependent variables are the effects.
- **Dependent Variable**
  A variable whose value depends on the value of the independent variable is called a dependent variable.
- **Function**
  When two variables are so related that one is dependent and another is independent, then the dependent variable is known as function of independent variable. For example, let us consider two variables x and y, which are related by the equation $y = 4x + 6$. We see that if we take $x = 1$, then we get $y = 10$; if we take $x = 0$, we get $y = 6$ and thus we see here that x is independent variable and y is dependent variable. So we may say that y is the function of x which is denoted by the symbol, $y = f(x)$. Hence we may conclude that any expression containing a variable is called a function of that variable. Thus *(i)* $ax + 10$, *(ii)* $2x^2 - 5x + 2$, *(iii)* $t^2 - 1$ and *(iv)* $e^t - 5$, where the expressions *(i)* and *(ii)* are functions of x and the expressions *(iii)* and *(iv)* are functions of t. The related variable on which the value of the function depends is also known as argument of the function.

Type of Functions
The different types of functions have been discussed as under:

a) **Linear Function:** A linear function represents a relationship between two variables, i.e., one dependent variable and another independent variable. Generally the functional form of the linear function is, $f(x) = ax + b$

   where, $f(x)$ is the dependent variable
   x is the independent variable
   b is the value of the dependent variable when x is zero.
   a is the coefficient of the independent variable.

   The above symbol $f(x)$ is read as *"function of x",* which represents the values of the dependent variable, and x represents values of the independent variable; $f(x)$ varies according to the rule of the function as x varies. For a linear function, the rule of the function states that *'a'* is to be multiplied by x and this product is to be added to b. This sum determines the value of the dependent variable $f(x)$.

b) Quadratic Function: The quadratic function is a second degree function which has important applications in business and economics. The general form of the quadratic function is, $f(x) = ax^2 + bx + c$

where, $f(x)$ is the dependent variable
x is the independent variable
a, b and c are the parameters of the functions.

The shape of the quadratic function is determined by the magnitude and signs of the parameters a, b and c.

c) Polynomial Functions: Linear and quadratic functions belong to the class of functions termed polynomial functions. The general form of the polynomial function is

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$

where, $a_0, a_1, a_2, a_3, \ldots a_n$ are parameters and n is a positive integer.

The parameters may be positive, negative or zero. The polynomial function is linear if $n=1$ and quadratic if $n=2$. This can be verified by comparing this for $n=1$ with the general form of the linear function, and $n=2$ with the general from of the quadratic function.

A polynomial function in which the largest exponent is $n=3$ is termed as cubic function. The general form of the cubic function is $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

d) Multivariate Functions: Functions in which the single dependent variable is related to more than one independent variable are termed as multivariate functions. The general form of multivariate function is, $f(x_1, x_2) = 2x_1+5x_1 x_2+6x_2$

where $f(x_1, x_2)$ is the dependent variable
x_1 is an independent variable
and x_2 is a second independent variable.

e) Exponential Functions: The exponential function is a specific function in which a constant is raised to a variable power rather than a variable raised to constant power. This function with a variable power is called the exponential function. The general form of exponential function is ,$h(x) = k.a^{f(x)}$

where $'a'$ is a constant greater than zero and not equal to one
and $f(x)$ is any real number function.

The domain of this function is the set of all real numbers, x, for which $f(x)$ is defined.

The exponential function states the constant rates of growth. As the independent variable increases by a constant amount in the exponential function, the dependent variable increases or decreases by a constant percentage. Hence, the value of an investment that increases by a constant percentage each period, the sales of a company that increase at a constant rate each period, and the value of an asset that declines at a constant rate each period are examples of functional relationship that are described by the exponential functions.

f) Logarithmic Functions: The inverse of the exponential function is the logarithmic function. The general form of the logarithmic function is, $y = log_a x$

where, y is the dependent variable
x is the independent variable
and $'a'$ is a constant, termed the base, that is greater than 0 and not equal to 1.

The logarithmic function arises when we ask the question, for what value of y is $a^y = x$.

If $a^y = x$, then $log_a x = y$ and vice-versa. Thus, the exponential function is corresponding to the logarithmic function, $y = log_a x$ is $a^y = x$.

Rate of Change

The rate of change of a function is the change in the value of the dependent variable with respect to the change in the value of the independent variable, i.e.,

$$\text{Rate of change} = \frac{\Delta f(x)}{\Delta x} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}}$$

If the independent variable x increases by Δx, the new value of the independent variable is $(x + \Delta x)$. For a linear function when $f(x) = ax + b$, the new value of the dependent variable for change in x is $f(x + \Delta x) = a(x + \Delta x) + b$.

To determine the amount of change in the dependent variable as the independent variable changes by Δx, the old value of the dependent variable, $f(x)$ is subtracted from the new value of the dependent variable, $f(x + \Delta x)$. That is, for the linear case,

$$\Delta f(x) = f(x + \Delta x) - f(x) = a(x + \Delta x) + b - (ax + b)$$
$$= (ax + a\Delta x + b - ax - b) = a.\Delta x.$$

Hence for the linear case, the rate of change is $\frac{\Delta f(x)}{\Delta x} = \frac{a.\Delta x}{\Delta x} = a$, which is the slope. The rate of change can be calculated for any function, linear or non-linear, using the same formula.

Notations for Functions

It 'x' is a variable of a function, then it is expressed as $f(x)$, $F(x)$, $g(x)$, ... $f_1(x)$, $f_2(x)$ ... which are basically called *functions of x*. Similarly it may be expressed as *'the f function of x'*, *'the F function of x'* ... etc.

Again, if more than one variable (x, y, z) exist in a particular function, it can be expressed as $f(x, y)$, $F(x, y, z)$. It is termed as *'the function of x and y'*, *'the F function of x, y, and z'* etc.

For example, If $f(x) = 2x^3 - 5x + 3$ and $F(x, y) = 3x^c + 5e^y - 3xy$,

then, $f(p) = 2p^3 - 5p + 3$ and $F(b, d) = 3b^c + 3e^d - 3bd.$

If the value of 'x' exists between a and$b$ then it is termed as domain or interval. If the interval is $a \leq x \leq b$, then it is called closed domain in which the values of a and$b$ are included.

Again if the interval is $a < x < b$, then it is called open domain, where the mid values of a and$b$ are included only. The samples of functions are presented as under:

$f(x) = 3x + 5$ → It is a linear function

$f(x) = 3x^2 - 3x + 8$ → It is a quadratic function

$f(x) = 4x^3 - 9x^2 + 3x - 6$ →It is a cubic function.

The following examples illustrate the use of functions

Example-1:

If $p(q) = q^2 - r^2 + 5$ and $h(r) = q^2 - r^2 + 5$; what is (i) $p(2)$ and (ii) $h(3)$?

Solution:

(a) We are given, $p(q) = q^2 - r^2 + 5$

$$\therefore p(2) = (2^2 - r^2 + 5) = 9 - r^2$$

(b) We are given, $h(r) = q^2 - r^2 + 5$

$$\therefore h(3) = (q^2 - 3^2 + 5) = q^2 - 4$$

Example-2:

Find $g(64)$, If $g(x) = \frac{x^{\frac{3}{2}}}{32} - 16x^{-\frac{1}{2}} + 2x^{\frac{1}{3}}$

Solution:

We are given, $g(x) = \frac{x^{\frac{3}{2}}}{32} - 16x^{-\frac{1}{2}} + 2x^{\frac{1}{3}}$

$$\therefore g(64) = \left[\frac{64^{\frac{3}{2}}}{32} - 16(64)^{-\frac{1}{2}} + 2(64)^{\frac{1}{3}}\right] = (16 - 2 + 8) = 28$$

Example-3:

Find (i) $g(a) - g(x - a)$, if $g(x) = x^2 + 10$

(ii) $f(x + a) - f(x)$; if $f(x) = x^2 - 3$

Solution:

(i) We are given $g(x) = x^2 + 10$

$\therefore g(a) - g(x - a) = (a^2 + 10) - \{(x - a)^2 + 10\}$

$= a^2 + 10 - x^2 + 2xa - a^2 - 10 = 2xa - x^2$

(ii) We are given, $f(x) = x^2 - 3$

$\therefore f(x + a) - f(x) = (x + a)^2 - 3 - (x^2 - 3)$

$= x^2 + 2xa + a^2 - 3 - x^2 + 3 = 2xa + a^2$

Example-4:

If $f(x) = \frac{ax+b}{bx+a}$, prove that $f(x).\, f(\frac{1}{x}) = 1$

Solution:

We have $f(x) = \frac{ax+b}{bx+a}$

Replacing x by $\frac{1}{x}$ on both sides, we get

$$f(\frac{1}{x}) = \frac{a.\frac{1}{x}+b}{b.\frac{1}{x}+a} = \frac{a+bx}{b+ax}$$

$\therefore f(x).f(\frac{1}{x}) = \frac{ax+b}{bx+a} \times \frac{a+bx}{b+ax} = 1$ (Proved)

Example-5:

Find the domain of the following function $\frac{x^2 + x + 5}{x^2 - 6x + 8}$

Solution:

Let $f(x) = \frac{x^2 + x + 5}{x^2 - 6x + 8}$

Clearly, $f(x)$ will be undefined if

$x^2 - 6x + 8 = 0$

or, $x^2 - 4x - 2x + 8 = 0$

or, $x(x - 4) - 2(x - 4) = 0$

or, $(x - 4)(x - 2) = 0$

$\therefore x = 2$ or $x = 4$

Hence, the domain of the definition of $f(x)$ is:

$-a < x < a$, but $x \neq 2$ and $x \neq 4$.

Example-6:

If $e^y - e^{-y} = 2x$, express y as an explicit function of x.

Solution:

We have $e^y - e^{-y} = 2x$ (Let $z = e^y$)

or, $z - \frac{1}{z} = 2x$

or, $z^2 - 1 = 2zx$

or, $z^2 - 2xz - 1 = 0$

$\therefore$ z = $\dfrac{2x \pm \sqrt{4x^2 - 4.1.(-1)}}{2.1}$

or, $e^y = \dfrac{2x \pm 2\sqrt{x^2+1}}{2}$

or, $e^y = x \pm \sqrt{x^2+1}$

or, $\log_e e^y = \log_e (x \pm \sqrt{x^2+1})$

or, $y = \log_e (x \pm \sqrt{x^2+1})$, which expresses y as an explicit function of x.

Example-7:

Find the range of the function $\dfrac{x}{1+x^2}$

Solution:

Let $y = \dfrac{x}{1+x^2}$

or, $x^2y + y = x$

or, $x^2y - x + y = 0$

$\therefore x = \dfrac{1+\sqrt{1-4y^2}}{2y}$

Since x is finite and real, we have

$y \neq 0$, and $1 - 4y^2 \geq 0$

or, $(1-2y)(1+2y) \geq 0$

$\therefore -\frac{1}{2} \leq y \leq \frac{1}{2}$

Therefore, the required range of the given function is: $-\dfrac{1}{2} \leq y \leq \dfrac{1}{2}$ and $y \neq 0$

or, $-\frac{1}{2} \leq$ y < 0 and 0 < y $\leq \frac{1}{2}$.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. What do you mean by constant and variable?
2. Define a function. What do you mean by domain interval and range of a function?
3. If $f(x-1) = 7x - 5$, find $f(x)$ and $f(x+2)$
4. If $f(x) = x^2 - x$, then prove that $f(h+1) = f(-h)$
5. If $f(x) = \dfrac{1}{x^2}$ show that $f(x+h) - f(x-h) = -\dfrac{4xh}{(x^2-h^2)^2}$
6. If $f(x) = \log_e \dfrac{1+x}{1-x}$, show that $f\left(\dfrac{2x}{1+x^2}\right) = 2f(x)$
7. If $y = f(x) = \dfrac{x-3}{2x+1}$ and $z = f(y)$, express z as a function of x.
8. Find the domain of the following function $\dfrac{x-2}{x^2-3x+2}$
9. Find the range of the function $\dfrac{x^2}{1+x^2}$

Lesson-2: Limit

After studying this lesson, you should be able to:

- Discuss the nature of fundamental theorems on limit;
- Apply the different methods of evaluating the limit.

Introduction

The concept of limit is an operation, which determines whether the value of a function exists in the neighborhood of a point at which the function is undefined. It is completely new concept in mathematics and is considered to be the basis of calculus. Now-a-days, this concept has wide application in the theoretical discussion in different branches of science including mathematics and in the solution of different problems in economics. In this lesson we shall make a brief discussion on the limit of a function and the application of fundamental theorems in evaluating limit of a function.

Limit of a Function

Generally, we are concerned with what happens to the value of the dependent variable $f(x)$ as the value of the independent variable x approaches some constant a. For example, the function $f$defined by $f(x) = x+2$ and notice what happens to the value of $f(x)$ as the value of x moves closer and closer to 2.

Let us set up a table of x and corresponding $f(x)$ values, as

x : 1.9 → 1.99 →1.999 →1.9999 2.0001 → 2.001 → 2.01 → 2.1

$f(x)$: 3.9 → 3.99 → 3.999 → 3.9999 4.0001 → 4.001 → 4.01 → 4.1

Fundamental Theorems of Evaluating Limit of a Function

The following theorems are most useful in the evaluation of limits.

For any real number a, assuming that $\lim_{x\to a} f(x)$ and lim $\lim_{x\to a} g(x)$ exist:

1. For any real constant k, $\lim_{x\to a} k = k$

2. For any real number n, $\lim_{x\to a} x^n = a^n$

3. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ in the root is defined

4. $\lim_{x\to a} k.f(x) = k.\lim_{x\to a} f(x)$

5. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

6. $\lim_{x\to a} f(x).g(x) = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$

7. $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} f(x) / \lim_{x\to a} g(x)$; if $\lim_{x\to a} g(x) \neq 0$

8. If n is any positive integer, then

   (a) $\lim_{x\to \propto^+} \frac{1}{x^n} = 0$ (b) $\lim_{x\to \propto^-} \frac{1}{x^n} = 0$

9. If n is any positive integer, then

   $\lim_{x\to 0^+} \frac{1}{x^n} = +\propto$ and $\lim_{x\to 0^+} \frac{1}{x^n} = +\propto$ (if n is even) or $-\propto$ (if n is odd)

10. $\lim_{x\to a} \log f(x) = \log \lim_{x\to a} f(x)$

11. $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$

The following examples illustrate the uses of these theorems.

Example-1:

Compute $\lim_{x \to 0} \frac{\sqrt{3+x} - \sqrt{3-x}}{x}$

Solution:

$$\lim_{x \to 0} \frac{\sqrt{3+x} - \sqrt{3-x}}{x} \times \frac{\sqrt{3+x} + \sqrt{3-x}}{\sqrt{3+x} + \sqrt{3-x}}$$

$$= \lim_{x \to 0} \frac{(3+x)-(3-x)}{x.\left[\sqrt{3+x} + \sqrt{3-x}\right]} = \lim_{x \to 0} \frac{2}{\sqrt{3+x} + \sqrt{3-x}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Example-2:

Evaluate $\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$; where, $g(x) = 7x + 9$

Solution:

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[7(x+h)+9\right] - \left[7x+9\right]}{h}$$

$$= \lim_{h \to 0} \frac{7x+7h+9-7x-9}{h} = \lim_{h \to 0} \frac{7h}{h}$$

$= \lim_{h \to 0} 7 = 7.$ The constant function 7 is continuous.

Example-3:

Evaluate $\lim_{x \to 0} \frac{e^x - e^{-x}}{x}$

Solution:

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{x}$$

$$= \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - .....\right)}{x}$$

$$= \lim_{x \to 0} \frac{2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ...\right)}{x}$$

$$= \lim_{x \to 0} 2\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + ...\right) = \lim_{x \to 0} (2 \times 1) = 2$$

Example-4:

Prove that $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Solution:

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

$$= \lim_{x\to 0} \frac{2\sin^2 \frac{x}{2}}{x} = \lim_{x\to 0} \frac{Sin2\frac{x}{2}}{\frac{x}{2}}$$

$$= \lim_{x\to 0} \frac{\sin^2 \frac{x}{2}}{\frac{x}{2}} \times \lim_{x\to 0} \sin\frac{x}{2} = (1 \times 0) = 0$$

Example-5:

Find (a) $\lim_{x\to 2}(3x^2 - x + 6)$ (b) $\lim_{x\to 3}(2x^2+1)(3x-4)$

Solution:

(a) $\lim_{x\to 2}(3x^2 - x + 6)$

$= \lim_{x\to 2} 3x^2 - \lim_{x\to 2} x + \lim_{x\to 2} 6$

$= [3(2)^2 - 2 + 6] = (12 - 2 + 6) = 16$

(b) $\lim_{x\to 3}(2x^2+1)(3x-4)$

$= [\lim_{x\to 3} 2x^2 + \lim_{x\to 3} 1]\,[\lim_{x\to 3} 3x - \lim_{x\to 3} 4]$

$= [2.(3)^2 + 1].\,[3.(3) - 4]$

$= [(18+1).(5)] = (19 \times 5) = 95$

Example-6:

$$\lim_{x\to 2} \frac{x^2-4}{x-2}$$

Solution:

Here Substituting $x = 2$, we get $\frac{0}{0}$ which does not exist.

Hence by rationalizing, $\lim_{x\to 2} \frac{x^2-4}{x-2}$

$= \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)}$ [as $x \neq 2$; $\therefore$ $x - 2 \neq 0$]

$= \lim_{x\to 2} (x+2) = (2 + 2) = 4$

$\therefore \lim_{x\to 2} \frac{x^2-4}{x-2} = 4.$

Some Important Limits

The following formulae are also used for evaluating the limit of a function.

(1) $\lim_{x\to 0} \frac{x^n - a^n}{x-a} = n.a^{n-1}$

(2) $\lim_{n\to 0} (1+n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$

(3) $\lim_{n\to 0} \frac{(1+x)^n - 1}{x} = n$

(4) $\lim_{n\to 0} \frac{\sin x}{x} = 1$

Example-7:

Show that, $\lim_{x\to 2} \left(x^2 - 3x + 5\right) = 3$

Solution:

We have $\lim_{x\to 2} \left(x^2 - 3x + 5\right) = 3$

$= \lim_{x\to 2} x^2 - \lim_{x\to 2} 3x + \lim_{x\to 2} 5$

$= \lim_{x\to 2} x . \lim_{x\to 2} x - 3 \lim_{x\to 2} x + 5$

= (2 × 2 – 3 × 2 + 5) = (4 – 6 + 5) = 3 (Proved)

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Give the definition of limits of a function.
2. Mention the fundamental theorems of evaluating a function.
3. Find $\lim_{x\to 2} (3x^2 + 2)$
4. If $f(x) = \frac{1}{x}$, find $\lim_{x\to 0} f(x)$
5. Evaluate $\lim_{x\to 1} \frac{x^2 - 3x + 2}{x - 1}$
6. Evaluate $\lim_{x\to\infty} \frac{2x+3}{x+1}$
7. Evaluate $\lim_{x\to 0} \left\{\frac{\sqrt{1+x} - \sqrt{1-x}}{x}\right\}$
8. Prove that $\lim_{x\to 4} \log\left(2x^{\frac{3}{2}} - 3x^{\frac{1}{2}} - 1\right) = 2 \log 3.$
9. Evaluate $\lim_{x\to -1} \frac{2x^2 - x - 3}{x^2 - 2x - 3}$
10. Find the value of $\lim_{x\to 0} \frac{\sqrt{1+2x} - \sqrt{1-3x}}{x}$

Lesson-3: Continuity

After studying this lesson, you should be able to:

- *Discuss the nature of continuity of a function;*
- *Apply the conditions for continuity of a function.*

Nature of Continuity

A function $f(x)$ is said to be continuous in an open or closed interval if it is continuous at all points in the interval. *For example*, the function $f(x) = x^2$ is continuous in the closed interval $-4 \leq x \leq 3$ when it is continuous at every point in the interval.

If the function $f(x)$ is not continuous at $x = a$, we say that the function $f(x)$ is discontinuous at $x = a$ and the point $x = a$ is called a point of discontinuity of the function. The function $f(x)$ is said to be discontinuous at $x = a$ if,

(i) $f(a)$ is undefined i.e. $f(x)$ does not possess a definite finite value at $x = a$

or, (ii) $\lim_{x \to a} f(x)$ does not exist

or, (iii) $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} f(x) \neq f(a)$

Continuity of a Function

The important concept of continuity of a function is developed from the theory of limit. A function 'f' is continuous at $x = a$ if and only if all of the following conditions apply to f at a.

1) $f(a)$ is defined, i.e., the domain of f includes $x = a;$
2) $\lim_{x \to a} f(x)$ exists;
3) $\lim_{x \to a} f(x) = f(a)$, whether x approaches to a from the left or from the right.

Continuity at an Interval

If a and$b$ are real numbers and $a < b$, then the set $\{x \mid a < x < b\}$ is called an open interval and is denoted by $[a, b]$. The set $\{x \mid a \leq x \leq b\}$ is called a closed interval and denoted by (a, b). The half-open interval $\{x \mid a \leq x < b\}$ is symbolized (a, b] whereas the half-closed interval $\{x \mid a < x \leq b\}$ is symbolized [a, b). In each case a and$b$ are the endpoints of the interval, and any x value such that $a < x < b$ is interior point.

A function f is continuous on an open interval if it is continuous at each number in that interval.

A function f is continuous on a closed interval (a, b) provided the following conditions are satisfied:

1. f is continuous over the open interval (a, b)
2. $f(x) \to f(a)$ as $x \to a$ from within (a, b)
3. $f(x) \to f(b)$ as $x \to b$ from within (a, b)

The following examples illustrate the requirements/conditions for continuity of a function.

Example-1:

Show that $f(x) = \dfrac{x^2 - 4}{x - 2}$ is not continuous $x = 2$ but continuous at $x = 3$.

Solution:

The conditions to be satisfied by a function before we can say that it is continuous at a particular point say $x = a$ are: $f(a)$, $\lim_{x \to a-} f(x)$ and $\lim_{x \to a+} f(x)$ should have definite and finite values and these are all equal.

Let us examine whether these conditions are satisfied by $f(x) = \dfrac{x^2 - 4}{x - 2}$ for $x = 2$.

Here $x = 2$, therefore we have

(i) $f(2) = \frac{2^2-4}{2-2} = \frac{0}{0}$, which is undefined.

Again by the method of finding the left hand and right hand side limits, we have

(ii) $\lim\limits_{x\to 2^-} \frac{x^2-4}{x-2} = \lim\limits_{x\to 2^-} \frac{(x+2)(x-2)}{(x-2)}$

$= \lim\limits_{x\to 2^-} (x+2)$

$= \lim\limits_{h\to 0} (2-h+2) = 4$

$\therefore$ L.H.S. limit = 4.

Again, $\lim\limits_{x\to 2^+} \frac{x^2-4}{x-2} = \lim\limits_{x\to 2^+} \frac{(x+2)(x-2)}{(x-2)}$

$= \lim\limits_{x\to 2^+} (x+2)$

$= \lim\limits_{h\to 0} (2+h+2) = 4$

$\therefore$ R.H.S. limit = 4.

Here, $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) \neq f(2)$

$\therefore f(x) = \frac{x^2-4}{x-2}$ is not continuous at $x = 2$.

Now, for $x = 3$,

i) $f(3) = \frac{3^2-4}{3-2} = 5$ and

ii) $\lim\limits_{x\to 3^-} \frac{x^2-4}{x-2} = \lim\limits_{x\to 3^-} \frac{(x+2)(x-2)}{(x-2)}$

$= \lim\limits_{h\to 0} (3-h+2) = 5$

$\therefore$ L.H.S. limit = 5.

Again, $\lim\limits_{x\to 3^+} \frac{x^2-4}{x-2} = \lim\limits_{x\to 3^+} \frac{(x+2)(x-2)}{(x-2)}$

$= \lim\limits_{h\to 0} (3+h+2) = 5$

$\therefore$ R.H.S. limit = 5.

Here, $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) \neq f(3)$

$\therefore f(x) = \frac{x^2-4}{x-2}$ is continuous at $x = 3$.

Example-2:

Show that $f(x) = 3x^2 + 2x - 1$ is continuous at $x = 2$. Also prove that $f(x)$ is continuous for all values of x.

Solution:

The condition is to be satisfied by a function if we can say that it is continuous at a particular point say $x = a$, where $\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$

Let us examine whether these conditions are satisfied by $f(x)= 3x^2 + 2x - 1$ for $x = 2$.

Here $a = 2$, therefore, we have (i) $f(2) = (3.\ 2^2 + 2.2 - 1) = 15$.

Again by the method of finding the left and right hand side limit, we have

(ii) $\lim_{x \to 2^-} (3x^2 + 2x - 1) = \lim_{h \to 0} \{(3(2 - h)^2 + 2(2 - h) - 1\} = 15$

∴ L.H.S. limit = 15.

Again (iii) $\lim_{x \to 2^+} (3x^2 + 2x - 1) = \lim_{h \to 0} \{(3(2 + h)^2 + 2(2 + h) - 1\} = 15$

∴ R.H.S. limit = 15.

We find the values of the function at $x = 2$, the LHS limit and RHS limit and all of these exist and finite and equal. Thus the $f(x) = 3x^2 + 2x - 1$ is continuous at $x = 2$.

We shall show further that $f(x) = 3x^2 + 2x - 1$ is continuous for all values of x.

Let $x = k$ be any value of x arbitrarily selected and find out whether given function is continuous at $x = k$

Here $a = k$, therefore $f(k) = 3k^2 + 2k - 1$ (finite number) .... (1)

Also, $\lim_{x \to k^-} (3x^2 + 2x - 1) = \lim_{h \to 0} \{(3(k - h)^2 + 2(k - h) - 1\}$

$= \lim_{h \to 0} (3k^2 - 6kh + 3h^2 - 2k + 2h - 1)$

$= (3k^2 + 2k - 1)$........(2)

Similarly we find that, $\lim_{x \to k^+} (3x^2 + 2x - 1) = 3k^2 + 2k - 1$ .... (3)

From (1), (2) and (3) we deduce that the given function is continuous at $x = k$.

Since k is any arbitrary value of x, therefore, $f(x)$ is continuous for all values of x.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Define the continuity of $f(x)$ at $x = a$. When is the function said to be continuous in the closed interval $a \leq x \leq b$?
2. Define the discontinuity of $f(x)$ at $x = a$.
3. Indicate the points of discontinuity of the function: $\frac{2x^2+6x-5}{12x^2+x-20}$.
4. The function $f(x) = \frac{x^3-8}{x^2-4}$ is undefined at $x = 2$. Redefine the function so as to make it continuous at $x = 2$.
5. It $f(x) = \frac{x^2-9}{x-3}$ when $x \neq 3$; state the value of $f(3)$ so that $f(x)$ is continuous at $x = 3$.

DIFFERENTIATION AND ITS USES IN BUSINESS PROBLEMS

9

Unit Highlights

- Lesson – 1: Differentiation
- Lesson – 2: Differentiation of Multivariate Functions

Technologies Used for Content Delivery

- BOUTUBE
- BOU LMS
- WebTV
- Web Radio
- Mobile Technology with MicroSD Card
- LP+ Office 365
- BTV Program
- Bangladesh Betar Program

Lesson-1: Differentiation

After studying this lesson, you should be able to:

- Explain the nature of differentiation;
- State the nature of the derivative of a function;
- State some standard formula for differentiation;
- Apply the formula of differentiation to solve business problems.

INTRODUCTION

Calculus is the most important ramification of mathematics. The present and potential managers of the contemporary world make extensive uses of this mathematical technique for making pregnant decisions. Calculus is inevitably indispensable to measure the degree of changes relating to different managerial issues. Calculus makes it possible for the enthusiastic and ambitious executives to determine the relationship of different variables on sound footings. Calculus in concerned with dynamic situations, such as how fast production levels are increasing, or how rapidly interest is accruing.

The term calculus is primarily related to arithmetic or probability concept. Mathematics resolved calculus into two parts - differential calculus and integral calculus. Calculus mainly deals with the rate of changes in a dependent variable with respect to the corresponding change in independent variables. Differential calculus is concerned with the average rate of changes, whereas Integral calculus, by its very nature, considers the total rate of changes in variables.

Differentiation

Differentiation is one of the most important operations in calculus. Its theory solely depends on the concepts of limit and continuity of functions. This operation assumes a small change in the value of dependent variable for small change in the value of independent variable. In fact, the techniques of differentiation of a function deal with the rate at which the dependent variable changes with respect to the independent variable. This rate of change is measured by a quantity known as derivative or differential co-efficient of the function. Differentiation is the process of finding out the derivatives of a continuous function i.e., it is the process of finding the differential co-efficient of a function.

Derivative of a Function

The derivative of a function is its instantaneous rate of change. Derivative is the small changes in the dependent variable with respect to a very small change in independent variable.

Let $y = f(x)$, derivative i.e. $\frac{dy}{dx}$ means rate of change in variable y with respect to change in variable x.

The derivative has many applications, and is extremely useful in optimization- that is, in making quantities as large (for example profit) or as small (for example, average cost) as possible.

Some Standard Formula for Differentiation

Following are the some standard formula of derivatives by means of which we can easily find the derivatives of algebraic, logarithmic and exponential functions. These are :

1. $\frac{dc}{dx} = 0$, where C is a constant.
2. $\frac{dx^n}{dx} = \frac{d}{dx}[x^n] = n.\,x^{n-1}$
3. $\frac{d}{dx}a.f(x) = a\frac{d}{dx}[f(x)]$
4. $\frac{d}{dx}\left(x^{-\frac{1}{n}}\right) = -\frac{1}{n}.x^{\frac{-(n+1)}{n}}$

5. $\frac{de^x}{dx} = \frac{d}{dx}\left(e^x\right) = e^x$

6. $\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}.\frac{d}{dx}\left[g(x)\right]$

7. If $y = f(u)$ and $U = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

8. $\frac{d}{dx}\left(a^x\right) = a^x.\log_e a$

9. $\frac{d\left[f(x) \pm g(x)\right]}{dx} = \frac{d\left[f(x)\right]}{dx} \pm \frac{d\left[g(x)\right]}{dx}$

10. $\frac{d}{dx}\left(\log_e x\right) = \frac{d}{dx}\left(\ln \quad x\right) = \frac{1}{x}$

11. If $Y = [f(x)]^n$ then, $\frac{dy}{dx} = n\ [f(x)]^{n-1}\ .\frac{d\left[f(x)\right]}{dx}$

12. $\frac{d}{dx}\log_a x = \frac{1}{x}$ $\log_a e$

13. $\frac{d\left[f(x).g(x)\right]}{dx} = f(x)\frac{d\left[g(x)\right]}{dx} + g(x)\frac{d\left[f(x)\right]}{dx}$

14. $\frac{d\left[\frac{f(x)}{g(x)}\right]}{dx} = \frac{g(x)\frac{d\left[f(x)\right]}{dx} - f(x)\frac{d\left[g(x)\right]}{dx}}{\left[g(x)\right]^2}$

15. $\frac{d}{dx}a^{g(x)} = a^{g(x)}.\frac{d}{dx}\left[g(x)\right].\log_a e$

16. If $U = f(x, y)$, $\frac{du}{dx} = \left[\frac{f(x + dx.y) - f(x, y)}{dx}\right]$ and $\frac{du}{dy} = \left[\frac{f(x, y + dy) - f(x, y)}{dy}\right]$

17. If y = e^{ax}, then its first derivative is equal to $\frac{de^{ax}}{dx}$ = e^{ax}

Second derivative is equal to $\frac{d^2e^{as}}{dx^2}$ = a^2e^{ax}

Third derivative is equal to $\frac{d^3e^{ax}}{dx3}$ = $a^3e^{ax}$and the nth derivative is denoted by

$\frac{d^ne^{ax}}{dx^n}$ = a^ne^{ax}

Derivative of Trigonometrically Functions

18. $\frac{d}{dx}$(Sin x) = Cos x. $\frac{d}{dx}$ (Cos x) = –Sinx

19. $\frac{d}{dx}$(tan x) = Sec^2x. $\frac{d}{dx}$ (Cot x) = – $Cosec^2$x

20. $\frac{d}{dx}$(Sec *x*) = Sec *x*. tan *x*; $\frac{d}{dx}$(Cosec *x*) = – Cosec *x*. Cot *x*

21. $\frac{d}{dx}\left(Sin^{-1}x\right) = \frac{1}{\sqrt{1 - x^2}}$; $\frac{d}{dx}\left(Cos^{-1}x\right) = -\frac{1}{\sqrt{1 - x^2}}$

22. $\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$; $\frac{d}{dx}\left(Cot^{-1}x\right)=-\frac{1}{1+x^2}$

23. $\frac{d}{dx}\left(Sec^{-1}x\right)=\frac{1}{x.\sqrt{x^2-1}}$; $\frac{d}{dx}\left(Co\sec^{-1}x\right)=-\frac{1}{x\sqrt{x^2-1}}$

$\text{Sin}^2x + \text{Cos}^2x = 1$; $\tan x = \frac{\text{Sin x}}{\text{Cos x}}$

$\text{Sec}^2x - \tan^2x = 1$; $\text{Cot x} = \frac{\text{Cos x}}{\text{Sin x}}$

When x and y are separately expressed as the functions of a third variable in the equation of a curve is known as parameter. In such cases we can find $\frac{dy}{dx}$ without first eliminating the parameter as follows:

Thus, if x =Q (t), y = ψ (t)

Then, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Let us illustrate these different derivatives by the following examples.

Example – 1:

If y = f(x) = a; find $\frac{dx}{dy}$

Solution:

$\frac{dy}{dx} = \frac{d(a)}{dx}$ =0, since a is a constant, i.e.,'a' has got no relationship with variable x.

Example – 2:

Differentiate the following functions, with respect to x,

(i) $y = \sqrt{x}$, (ii) $y = 8x^{-5}$ (iii) $y= 3x^3 - 6x^2 + 2x - 8$

Solution:

We know that $\sqrt{x} = x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}.x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$. Hence $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

(ii) $\frac{dy}{dx} = \frac{d}{dx}(8x^{-5})$

$= 8\frac{d}{dx}(x^{-5}) = 8(-5)\,x^{-6} = -40x^{-6} = \frac{-40}{x^6}$.

Therefore, $\frac{dy}{dx} = \frac{-40}{x^6}$

(iii) $\frac{dy}{dx} = \frac{d}{dx}(3x^3 - 6x^2 + 2x - 8)$

$= \frac{d}{dx}(3x^3) - \frac{d}{dx}(6x^2) + \frac{d}{dx}(2x) - \frac{d}{dx}(8)$

$= 3.3x^{3-1} - 2.6.x^{2-1} + 2-0 = 9x^2 - 12x + 2$

Thus, $\frac{dy}{dx} = 9x^2 - 12x + 2.$

Example –3:

Differentiate $e^x(\log x) . (2x^2+3)$ with respect to x.

Solution:

Let $y = e^x.(\log x) . (2x^2 + 3)$

$$\frac{dy}{dx} = e^x (\log x)\frac{d}{dx}(2x^2+3) + e^x(2x^2+3).\frac{d}{dx}(\log x) + (\log x).(2x^2+3).\frac{d}{dx}(e^x)$$

$$= e^x(\log x)(4x) + e^x(2x^2+3)\frac{1}{x} + \log x(2x^2+3)e^x$$

$$= e^x[4x.\log x + \frac{2x^2+3}{x} + (2x^2+3)\log x]$$

$$\text{So, } \frac{dy}{dx} = e^x[4x.\log x + \frac{2x^2+3}{x} + (2x^2+3)\log x]$$

Example–4:

If $y = \frac{2+3\log x}{x^2+5}$, find $\frac{dy}{dx}$

Solution:

$$y = \frac{2+3\log x}{x^2+5}$$

$$\frac{dy}{dx} = \frac{(x^2+5)\frac{d}{dx}(2+3\log x) - (2+3\log x)\frac{d}{dx}(x^2+5)}{(x^2+5)^2}$$

$$= \frac{(x^2+5)(\frac{3}{x}) - (2+3\log x)(2x)}{(x^2+5)^2} = \frac{\frac{15}{x} - x - 6x\log x}{(x^2+5)^2}. \text{Thus, } \frac{dy}{dx} = \frac{\frac{15}{x} - x - 6x\log x}{(x^2+5)^2}$$

Example–5:

Find $\frac{dy}{dx}$, if $y = \log\sqrt{4x+3}$

Solution:

Let $y = \log\sqrt{4x+3}$

$$= \log(4x+3)^{\frac{1}{2}} = \frac{1}{2}\log(4x+3)$$

$$\frac{dy}{dx} = \frac{d}{dx}[\frac{1}{2}\log(4x+3)] = \frac{1}{2}.\frac{1}{4x+3}.\frac{d}{dx}(4x+3) = \frac{1}{2}.\frac{1}{4x+3}.4 = \frac{2}{4x+3}$$

$$\therefore \frac{dy}{dx} = \frac{2}{4x+3}$$

Example–6:

Find the first, second and third derivatives when $y = x.e^{x^2}$

Solution:

$y = x.e^{x^2}$

First derivative,

$$\frac{dy}{dx} = x\frac{d}{dx}\ e^{x^2} + e^{x^2}.\frac{d}{dx}(x) = x.e^{x^2}.\ 2x + e^{x^2}.1 = e^{x^2}\ (2x^2+1)$$

Second derivative, $\frac{d^2y}{dx^2} = e^{x^2}.2x(2x^2+1) + e^{x^2}.4x$

$$= e^{x^2}.\ 4x^3 + e^{x^2}\ .2x + e^{x^2}.\ 4x = e^{x^2}.\ (4x^3 + 2x + 4x)$$

$$= e^{x^2}.\ (4x^3 + 6x)$$

Third derivative,

$$\frac{d^3y}{dx^3} = e^{x^2}\ 2x\ (4x^3 + 6x) + e^{x^2}\ (12x^2 + 6)$$

$$= e^{x^2}.\ 8x^4 + e^{x^2}\ 12x^2 + e^{x^2}\ (12x^2 + 6) = e^{x^2}\ (8x^4 + 12x^2 + 12x^2 + 6)$$

$$= e^{x^2}\ (8x^4 + 24x^2 + 6)$$

Example–7:

If $y = x^{\log x}$, find $\frac{dy}{dx}$

Solution:

Given, $y = x^{\log x}$

Taking logarithm of both sides, we have

$\log y = \log (x^{\log x}) = \log x.\ \log x = (\log x)^2$

Differentiating with respect to x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}(\log x)^2 = 2\ (\log x)^{2-1}\frac{d}{dx}(\log x) = 2\ \log x.\ \frac{1}{x}$$

Hence, $\frac{dy}{dx} = y\ (2\ \log x.\ \frac{1}{x}) = \frac{2x^{\log x}.\ \log x}{x}$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions :

1. Define differentiation. What are the fundamental theorems of differentiation?
2. Why is the study of differentiation important in managerial decision making?
3. Find the derivative of the following functions with respect to x.

   i) $5x^4 + \frac{3}{x5} - 8x^2 + 7x$, (ii) $(2x^3 - 5x^{-2} + 2)\ (4x^2 - 3\sqrt{x})$ (iii) $\frac{5x^2+9}{3x-2}$

   (iv) $(3x^2 - 2x + 5)^{3/2}$ (v) $5e^x \log x$, (vi) $x^2 + 3xy + y^3 = 5$ (vii) e^{x^x}
4. If $y = x^3 \log n$, show that $\frac{d^4y}{dx^4} = \frac{6}{x}$
5. Differentiate the following w. r. to x

   (i) $\text{Sin } x^{\text{Cos } x}$ (ii) $\frac{\sin x}{\cos x}$ (iii) $e^{4x} + \log \sin x$

   iv) $(\sin^{-1}x)^{\log x}$
6. If $y = 8x^3 - 5x^{3/2} + 3x^2 - 7x + 5$: find $\frac{d^3y}{dx^3}$

Lesson-2: Differentiation of Multivariate Functions

After studying this lesson, you will be able to:

- State the nature of multivariate function;
- Explain the partial derivatives;
- Explain the higher- order derivatives of multivariate functions;
- Apply the techniques of multivariate function to solve the problems.

Introduction

The concept of the derivative extends directly to multivariate functions. During the discussion of differentiation, we defined the derivative of a function as the instantaneous rate of change of the function with respect to independent variable. In multivariate functions, there are more than one independent variable involved and thereby, the derivative of the function must be considered separately for each independent variable. For example, $z = f(x, y)$ is defined as a function of two independent variables if there exists one and only one value of z in the range of f for each ordered pair of real number (x, y) in the domain of f. By convention, z is the dependent variable; x and y are the independent variables.

To measure the effect of a change in a single independent variable (x or y) on the dependent variable (z) in a multivariate function, the partial derivative is needed. The partial derivative of z with respect of 'x' measures the instantaneous rate of change of z with respect to x while y is held constant. It is written $\frac{dz}{dx}, \frac{df}{dx}$, $f_x\ (x, y), f_x$ or Zx. The partial derivative of z with respect to y measures the rate of change of z with respect to y while x is held constant. It is written as: $\frac{dz}{dy}, \frac{df}{dy}$, $f_y\ (x, y), f_y$ or Zy.

Mathematically it can be expressed in the following way:

$$\frac{dz}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$\frac{dz}{dy} = \lim_{\Delta x \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Partial differentiation with respect to one of the independent variables follows the same rules as differentiation while the other independent variables are treated as constant.

This is illustrated by the following examples.

Example–1:

Find the partial derivatives of $M = f(x, y, z) = x^2 + 5y^2 + 20xy + 4z$.

Solution:

To determine the partial derivative of f with respect to x, we treat y and z as constants.

$$\frac{dm}{dx} = f_x' = 2x + 20y.$$

Similarly, in determining the partial derivative of f with respect to y, we treat x and z as a constants. $\frac{dm}{dy} = f_y' = 10y + 20x$

Finally, treating x and y as constants, we obtain the partial derivative of f with respect to z $\frac{dm}{dz} = f_z' = 4$

The same procedure is applied in the following examples.

Example – 2:

Determine the partial derivatives of

$Z = 5x^3 - 3x^2y^2 + 7y^5$

Solution:

$$\frac{dz}{dx} = \frac{d}{dx}(5x^3) - 3y^2\frac{d}{dx}(x^2) + \frac{d}{dx}(7y^5)$$

$$= 15x^2 - 6xy^2 + 0$$

$$\therefore \frac{dz}{dx} = 15x^2 - 6xy^2$$

Again $\frac{dz}{dy} = \frac{d}{dy}(5x^3) - 3x^2\frac{d}{dy}(y^2) + \frac{d}{dx}(7y^5)$

$$= 0 - 6x^2y + 35y^4$$

$$\therefore \frac{dz}{dy} = 35y^4 - 6x^2y.$$

Example–3:

Find the partial derivatives of $z = (5x + 3)(6x + 2y)$

Solution:

$$\frac{dz}{dx} = (5x + 3).\frac{d}{dx}(6x+2y) + (6x + 2y).\frac{d}{dx}(5x+3)$$

$$= (5x + 3).\ 6 + (6x + 2y).\ 5$$

$$= 30x + 18 + 30x + 10y$$

$$\therefore \frac{dz}{dx} = 60x + 10y + 18$$

Again $\frac{dz}{dy} = (5x + 3).\frac{d}{dy}(6x+2y) + (6x + 2y).\frac{d}{dx}(5x+3)$

$$= (5x + 3).\ 2 + (6x + 2y).\ 0$$

$$\therefore \frac{dz}{dy} = 10x + 6 + 0 = 10x + 6$$

Higher-Order derivatives of Multivariate Functions

The rules for determining higher-order derivatives of functions of one independent variable apply to multivariate functions. Derivatives of multivariate functions are taken with respect to one independent variable at a time, the remaining independent variables being considered as constants. The same procedure applies in determining higher-order derivatives of multivariate functions.

For instance, a function *f (x, y)* may have four second-order partial derivatives as follows:

Original function First partial derivatives Second-order partial derivatives

$f''_{xx}(x, y)$ or $\frac{d^2f}{dx\,dx}$ or $\frac{d^2 f}{dx^2}$

$f''_{y}(x, y)$ or $\frac{d(f)}{dx}$

$f''_{yx}(x, y)$ or $\frac{d^2f}{dy\,dx}$

$f(x, y)$

$f'_{y}(x, y)$ or $\frac{d(f)}{dy}$ $\qquad$ $f''_{yx}(x, y)$ or $\frac{d^2f}{dx\,dy}$

$f''_{yy}(x, y)$ or $\frac{d^2f}{dy\,dy}$ or $\frac{d^2 f}{dy^2}$

The second-order partial derivatives f''_{xx} and f''_{xy} are obtained by differentiating f''_x , with respect to x and with respect to y respectively. Similarly the second-order partial derivatives f''_{yx} and f''_{yy} are obtained by differentiating *fy*, with respect to x and with respect to y respectively.

The cross (or mixed) partial derivative f''_{xy} or f''_{yx} indicates that first the primitive function has been partially differentiated with respect to one independent variable and then that partial derivative has in turn been partially differentiated with respect to the other independent variable:

$$f''_{xy} = \left(f'_x\right)'_y = \frac{d}{dy}\left(\frac{dz}{dx}\right) = \frac{d^2z}{dydx}$$

$$f''_{yz} = \left(f'_y\right)'_x = \frac{d}{dx}\left(\frac{dz}{dy}\right) = \frac{d^2z}{dxdy}$$

The cross partial is a second-order derivative, which are equal always. That is

$$\frac{d^2z}{dydx} = \frac{d^2z}{dxdy}$$

$$or, f''_{xy} = f''_{yx}$$

This is illustrated by the following examples.

Example–4:

Determine (a) first, (b) second and (c) cross partial derivatives of $Z = 7x^3 + 9xy + 2y^5$.

Solution:

(a) $\frac{dz}{dx} = Z_x = 21x^2 + 9y;$ $\quad \frac{dz}{dy} = Z_y = 9x + 10y^4.$

(b) $\frac{d^2z}{dx^2} = Z_{xx} = 42x;$ $\quad \frac{d^2z}{dy^2} = Z_{yy} = 40y^3.$

(c) $\frac{d^2z}{dydx} = \frac{d}{dy} \cdot \frac{dz}{dx} = \frac{d}{dy}(21x^2+9y) = Z_{xy} = 9$

$\frac{d^2z}{dxdy} = \frac{d}{dx} \cdot \frac{dz}{dy} = \frac{d}{dx}(9x+10y^4) = Z_{yx} = 9.$

Example–5:

Determine all first and second-order derivatives of $Z = (x^2+3y^3)^4$

Solution:

$\frac{dz}{dx} = 4\,(x^2+3y^3)^3 \cdot 2x = 8x\,(x^2+3y^3)^3$

$\frac{dz}{dy} = 4\,(x^2+3y^3)^3 .\ 9y^2 = 36y^2\,(x^2+3y^3)^3$

$\frac{d^2z}{dx^2} = 8x[3(x^2+3x^3)^2\,(2x)] + (x^2+3y^3)^3 \cdot 8$

$= 48x^2\,[x^2+3x^3]^2 + 8(x^2+3y^3)^3$

$\frac{d^2z}{dy^2} = 36y^2\,[3(x^2+3y^3)^2\,(9y^2)] + (x^2+3y^3)^3\ 72y$

$\frac{d^2z}{dydx} = \frac{d}{dy} \cdot \frac{dz}{dx} = 8x\,[3(x^2+3y^3)^2\,(9y^2)]+0$

$= 216xy^2\,(x^2+3y^3)^2$

$\frac{d^2z}{dxdy} = \frac{d}{dx} \cdot \frac{dz}{dy} = 36y^2\,[3\,(x^2+3y^3)^2 .\ 2x]$

$= 216xy^2\,(x^2+3y^3)^2$

Optimization of Multivariate Functions

For a multivariate function such as $z = f(x, y)$ to be at a relative minimum or maximum, the following three conditions must be fullfiled /met:

1. Given $z = f(x, y)$, determine the first-order partial derivatives, $f'_x(x, y)$ and $f'_y(x, y)$ and all critical point/values (a, b); that is, determine all values (a, b) such that $f'_x(a, b) = f''_y(a, b) = 0$

2. Determine the second-order partial derivatives, $f''_{xx}(x,y), f''_{xy}(x,y), f''_{yx}(x,y)$ and $f''_{yy}(x,y)$

[Note: the cross partial derivatives $f''_{xy}(x,y)$ (x, y) and $f''_y(x,y)$ (x, y) must be equal to one another; otherwise the function is not continuous.]

3. Where (a, b) is a critical point on f, let $D = f''_{xx}$ (a, b). $f''_{yy}(a, b) - [f''_{xy}(a, b)]^2$

Then

(i) If D>0 and $f''_{xx}(a, b) < 0$, f has a relative maximum at (a, b)

(ii) If D>0 and $f''_{xx}(a, b) > 0$, f has a relative minimum at (a, b).

(iii) If D<0, f has neither a relative maximum nor a relative minimum at (a, b)

(iv) If D = 0, no conclusion can be drawn; further analysis is required.

This is illustrated by the following example.

Example–7:

Determine the critical points and specify whether the function had a relative maximum or minimum,

$$z = 2y^3 - x^3 + 147x - 54y + 12.$$

Solution:

By taking the first-order partial derivatives, setting them equal to zero, and solving for x and y:

	$z_x = -3x^2 + 147 = 0$	$z_y = 6y^2 - 54 = 0$
or,	$x^2 = 49$	or, $6y^2 = 54$
	$\therefore x = \pm 7$	$\therefore y = \pm 3$

This mean that we must investigate four critical points, namely (7, 3), (7, –3), (–7, 3) and (–7,–3).

The second-order partial derivatives are

$Z_{xx} = -6x$	$Z_{yy} = 12y$
(1) $Z_{xx}(7, 3) = -6(7) = -42 < 0$	$Z_{yy}(7, 3) = 12 \times 3 = 36 > 0$
(2) $Z_{xx}(7, -3) = -6(7) = -42 < 0$	$Z_{yy}(7, -3) = 12x{-3} = -26 < 0$
(3) $Z_{xx}(-7, 3) = -6(-7) = 42 > 0$	$Z_{yy}(-7, 3) = 12x3 = 36 > 0$
(4) $Z_{xx}(-7, -3) = -6(-7) = 42 > 0$	$Z_{yy}(-7, -3) = 12x{-3} = -36 < 0$

Since there are different signs for each of the second-order partials in (1) and (4), the function cannot be at a relative maximum or minimum at (7, 3) or (–7,–3). When f''_{xy} and f''_{yy} are of different signs, $(f''_{xx} . f''_{yy})$ cannot be greater than f''_{xy} and the function is at a saddle point.

With both signs of second-order partials negative in (2) and positive in (3), the function may be at a relative maximum at (7,–3) and at a relative minimum at (–7, 3), but the third condition must be tested first to ensure against the possibility of an inflection point.

From the first partial derivative, we obtain cross partial derivatives and check to make sure that $Z_{xx}(a, b), Z_{yy}(a, b) > [Z_{xy}(a, b)]^2$

Hence, $Z_{xy} = 0$ and $Z_{yx} = 0$

$Z_{xx}(a, b).Z_{yy}(a, b) > [Z_{xy}(a, b)]^2$

From (2), $(-42).(-36) > (0)^2$

□□ $\therefore 1512 > 0$

From (3), (42). (36) > $(0)^2$

$\therefore$ 1512 > 0.

Hence the function has a relative maximum at (7,−3) and a relative minimum at (−7, 3).

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Determine first, second and cross – partial derivatives of
   $f(x, y) = 2x^2 + 4xy^2 - 5y^2 + y^3$
2. Determine the first and second–order partial derivatives of the function
   $f(x, y, z) = x^2e^y$ In z.
3. Examine the function, $z(x, y) = x^2 + y^2 - 4x + 6y$ for relative maxima or minima by using second- order derivative test.
4. Find the partial derivatives for $z = (6x + 4)(4x + 2y)$.

MATRIX ALGEBRA

10

Unit Highlights

- Lesson – 1: Matrix: An Introduction
- Lesson – 2: Matrix Operations
- Lesson – 3: Determinant
- Lesson – 4: Matrix Inversion

Technologies Used for Content Delivery

- BOUTUBE
- BOU LMS
- WebTV
- Web Radio
- Mobile Technology with MicroSD Card
- LP+ Office 365
- BTV Program
- Bangladesh Betar Program

Lesson-1: Matrix: An Introduction

After studying this lesson, you should be able to:

- State the nature of a matrix;
- Explain matrix representation of data.
- Define different types of matrices.

Introduction

J. J. Sylvester was the first to use the word 'matrix' in 1850 and later on in 1858 Arthur Cayley developed the theory of matrices in a systematic way. Matrix is a powerful tool of modern mathematics and its study is becoming important day by day due to its wide applications in every branch of knowledge. Matrix arithmetic is basic to many of the tools of managerial decision analysis. It has an important role in modern techniques for quantitative analysis of business and economic decisions. The tool has also become quite significant in the functional business and economic areas of accounting, production, finance and marketing.

Matrix

Whenever one is dealing with data, there should be concern for organizing them in such a way that they are meaningful and can be readily identified. Summarizing data in a tabular form can serve this function. A matrix is a common device for summarizing and displaying numbers or data. Thus, a matrix is a rectangular array of elements and has no numerical value. The elements may be numbers, parameters or variables. The elements in horizontal lines are called rows, and the elements in vertical lines are called columns.

A matrix is characterized further by its dimension. The dimension or order indicates the number of rows and the number of columns contained within the matrix. If a matrix has *m* rows and *n* columns, it is said to have dimension ($m \times n$), which is read as: *m* by *n*.

Example: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Types of Matrices:

Row Matrix: The matrix with only one row is called a row matrix or row vector.

For example: $A = \begin{pmatrix} 2 & 3 & 4 \end{pmatrix}$.

Column Matrix: The matrix with only one column is called a column matrix or column vector.

For example: $A = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

Row matrix and column matrix are usually called as row vector and column vector respectively.

Square Matrix: If the number of rows and the number of columns of a matrix are equal then the matrix is of order $n \times n$ and is called a square matrix of order *n*.

For example: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Rectangular Matrix: If the number of rows and the number of columns of a matrix are not equal then the matrix is called a rectangular matrix.

For example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Singular matrix: A square matrix A is said to be singular if the determinant formed by its elements equal to zero.

For example: Let $A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$.

Determinant of $A = |A| = (2 \times 2) - (4 \times 1) = 0$.

Hence A is a singular matrix.

Non-singular Matrix:
A square matrix A is said to be non-singular if the determinant formed by its elements is non-zero.

For example: $A = \begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}$

$|A| = (5 \times 4) - (3 \times 2) = 20 - 6 = 14$.

Hence A is a non-singular matrix.

Null or Zero Matrix: The matrix with all of its elements equal to zero is called a null matrix or zero matrix.

For example: $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Diagonal Matrix: A matrix whose all elements are zero except those in the principal diagonal is called a diagonal matrix.

For example: $A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$

Scalar Matrix: A diagonal matrix, whose diagonal elements are equal, is called a scalar matrix.

For example: $A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

Sub-Matrix: A matrix that is obtained from a given matrix by deleting any number of rows and any number of columns is called a sub-matrix of the given matrix.

For example: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a sub-matrix of $B = \begin{pmatrix} 5 & 3 & 2 \\ 1 & 1 & 2 \\ 7 & 3 & 4 \end{pmatrix}$

Unit matrix or Identity matrix: A matrix with every element in the principal diagonal equals to one and all other elements equal to zero is called a unit matrix. A unit matrix is a square matrix. It is denoted by I.

For example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Equal Matrix: Two matrices A and B are said to be equal if their corresponding elements are equal.

For example: Let $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ then $A = B$

Transpose of a Matrix: If the columns of a given matrix A are changed into rows or the rows are changed into columns, the matrix thus formed is called the transpose of the matrix A and it is generally denoted by A^T.

For example: Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ then $A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$

Symmetric Matrix: A square matrix A is called symmetric if it be same as its transpose so that $A = A^T$.

For Example: Let $A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$ then $A^T = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$

i.e., $A = A^T$, so A is a symmetric matrix.

Skew-Symmetric Matrix: A square matrix A is called skew-symmetric if $A^T = -A$.

For example: Let $A = \begin{pmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{pmatrix}$

thenA^T = $A^T = \begin{pmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{pmatrix} = -A$

i.e., $A^T = -A$, hence A is a skew-symmetric matrix.

Involuntary Matrix: A square matrix A is called involuntary matrix provided it satisfies the relation $A^2 = I$, where I is the identity matrix.

For example: $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

Idempotent Matrix: A square matrix A is called idempotent matrix provided it satisfies the relation $A^2 = A$.

Example: $A = \begin{pmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$

Nilpotent Matrix: A square matrix A is called nilpotent matrix of order ***m*** provided it satisfies the relation $A^m = 0$ and $A^{m-1} \neq 0$, where m is a positive integer and 0 is the null matrix.

For example: $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ -1 & -2 & -5 \end{pmatrix}$ since $A \neq 0$, $A^2 = 0$

Complex Conjugate of a Matrix: It is a matrix obtained by replacing all its elements by their respective complex conjugates.

For example: If $A = \begin{pmatrix} 2 & +3i & 5 \\ 3 & -3i & 7 \end{pmatrix}$ then $\overline{A} = \begin{pmatrix} 2 & -3i & 5 \\ 3 & +3i & 7 \end{pmatrix}$

Hermitian Matrix: A matrix having complex elements of a square matrix A is a Hermitian matrix. If $(A)' = A$, then A is called Hermitian matrix.

Skew-Hermitian Matrix: A matrix having complex elements for matrix $A.(A) = -A.A$ is skew hermitian matrix.

Co-factor Matrix

A matrix, which is formed by the co-factors of the corresponding elements, is called co-factor matrix and is denoted by A^C.

For example: If a matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then, the co-factor matrix, $A^C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$

Adjoint Matrix:

The Adjoint matrix is the transpose of the co-factor matrix, that is $adjA = A_j = (cof\ A)^T$

Orthogonal Matrix: A square matrix A is called an orthogonal matrix if $AA^T = A^T A = I$, where I is an identity matrix and A^T is the transpose matrix of A.

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. What do you understand by matrix?
2. Why matrix algebra is so important in business and economics? Explain.
3. Discuss the various types of matrices.
4. In an examination, 20 students from college A, 30 students from college B and 40 students from college C appeared. Only 15 students from each college could get through the examination. Out of them 10 students from college A, 5 students from college B and 10 students from college C secured full marks. Write down the above data in matrix form.

Lesson-2: Matrix Operations

After studying this lesson, you should be able to:

- Express the concept of matrix operations;
- Add or subtract given matrices;
- Multiply given matrices.

Introduction

The operations of matrices are addition, subtraction, multiplication and division of which addition and multiplication are the main operations. In this lesson we will discuss some of the operations of matrix algebra.

Matrix Addition

Two matrices of the same dimensions are said to be conformable for addition. The addition is performed by adding corresponding elements from the two matrices and entering the result in the same row-column position of a new matrix.

If A and B are two matrices, each of size $m \times n$ then the sum of A and B is the $m \times n$ matrix C whose elements are $C_{ij} = A_{ij} + B_{ij}$; i = 1, 2, 3… m and j = 1, 2, 3, ….n.

Properties of Matrix Addition:

- *Commutative law*: Matrix addition is commutative. If A and B are two matrices of same order $m \times n$, then $A + B = B + A$.
- *Associative law*: Matrix addition is associative. If A, B and C are three matrices of same order $m \times n$, then $A + (B + C) = (A + B) + C$.
- *Distributive law*: If A and B are two matrices of same order $m \times n$, and K is any scalar, then $K(A + B) = KA + KB$.
- *Existence of additive identity*: If O denotes null matrix of the same order as that of A, then $A + O = A = O + A$.
- *Existence of an additive inverse*: If A be any given $m \times n$ matrix and there exists another $m \times n$ matrix B such that $A + B = O = B + A$; where O be the $m \times n$ null matrix.
- *Cancellation law*: If A, B and C are three matrices of same order $(m \times n)$, then $A + C = B + C \Rightarrow A = B$.

Example-1:

Find the sums $A + B$ of the following matrices

$A = \begin{pmatrix} 8 & 9 \\ 12 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 13 & 4 \\ 2 & 6 \end{pmatrix}$

Solution:

$$A + B = \begin{pmatrix} 8+13 & 9+4 \\ 12+2 & 7+6 \end{pmatrix} = \begin{pmatrix} 21 & 13 \\ 14 & 13 \end{pmatrix}$$

Matrix Subtraction

The subtraction of two matrices is possible only when they are of the same order. Such matrices are said to be conformable for subtraction. The subtraction is performed by subtracting corresponding elements of the two matrices and entering the result in the same row-column position of a new matrix.

If A and B are two matrices, each of size $m \times n$ then the subtraction of A and B is the $m \times n$ matrix C whose elements are $C_{ij} = A_{ij} - B_{ij}$; i = 1, 2, 3… m and j = 1, 2, 3, ….n.

Example-2:

Find the difference $A - B$ of the following matrices

$$A = \begin{pmatrix} 3 & 7 & 11 \\ 12 & 9 & 2 \end{pmatrix} \text{and } B = \begin{pmatrix} 6 & 8 & 1 \\ 9 & 5 & 8 \end{pmatrix}$$

Solution:

$$A - B = \begin{pmatrix} 3-6 & 7-8 & 11-1 \\ 12-9 & 9-5 & 2-8 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 10 \\ 3 & 4 & -6 \end{pmatrix}$$

Scalar Multiplication of a Matrix

A matrix can be multiplied by a constant by multiplying each component in the matrix by the constant. The result is a new matrix of the same dimensions as the original matrix.

If K is any real number and $A = [a_{ij}]$ is an $m \times n$ matrix, then the product KA is defined to be the matrix whose components are given by K times the corresponding component of A, i.e., $KA = [Ka_{ij}]$

Laws of scalar multiplication:

(i) $K(A+B) = KA + KB$

(ii) $(K_1 + K_2)A = K_1A + K_2A$

(iii) $IA = A$

(iv) $(K_1K_2)A = K_1(K_2A)$.

Example-3:

If $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$, Find $5A$.

Solution:

$$5A = 5\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 5 \\ 10 & 5 & 10 \\ 15 & 10 & 5 \end{pmatrix}$$

Multiplication of Matrices

If the number of columns of the first matrix is equal to the number of rows of the second matrix, such matrices are said to be conformable for multiplication. Let A be a matrix of order $m \times p$ and B be a matrix of order $p \times n$. Then the product AB is defined to be a matrix C of order $m \times n$.

Properties of Matrix Multiplication

- *Associative law*: Multiplication of matrices is associative i.e. $A(BC) = (AB)C$.
- *Distributive law*: Multiplication of matrices is distributive with respect to matrix addition i.e. $A(B+C) = AB + AC$.
- *Multiplication of a matrix by a null matrix:* If A is $n \times m$ and O is $m \times n$ matrices, then $AO = O = OA$.
- *Multiplication of a matrix by a unit matrix:* If A is a square matrix of order $n \times n$ and I is the unit matrix of same order, then $IA = A = AI$.
- *Multiplication of matrix by itself:* If A is a square matrix then $A.A = A^2$.

Example-4:

Find AB, where $A = [9 \quad 11 \quad 3]$ and $B = \begin{pmatrix} 2 \\ 6 \\ 7 \end{pmatrix}$

Solution:

The matrices *A* and *B* are conformable for multiplication. The dimensions of A and B are 1×3 and 3×1 respectively, i.e., the product matrix AB will be 1×1 and a scalar, derived by multiplying each element of the row vector by its corresponding element in the column vector and then summing the products.

$$AB = [(9 \times 2) + (11 \times 6) + (3 \times 7)] = 18 + 66 + 21 = 105.$$

Example-5:

If $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$.

Find (i) $3A - 4B$

(ii) $2A - 3B$

Solution:

(i) $3A - 4B = 3\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 9 & 3 \\ 0 & -3 & 15 \end{pmatrix} - \begin{pmatrix} 4 & 8 & -4 \\ 0 & -4 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 6-4 & 9-8 & 3-(-4) \\ 0-0 & -3-(-4) & 15-12 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 7 \\ 0 & 1 & 3 \end{pmatrix}$$

(ii) $2A - 3B = 2\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix} - 3\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 4 & 6 & 2 \\ 0 & -2 & 10 \end{pmatrix} - \begin{pmatrix} 3 & 6 & -3 \\ 0 & -3 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 4-3 & 6-6 & 2-(-3) \\ 0-0 & -2-(-3) & 10-9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \end{pmatrix}$$

Example-6:

If $A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$

then find *AB*. Whether *BA* exists? Give reason.

Solution:

$$AB = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3.1 + 1.2 + 2.1 & 3.4 + 1.2 + 2.0 \\ 0.1 + 1.2 + 1.1 & 0.4 + 1.2 + 1.0 \\ 1.1 + 2.2 + 0.1 & 1.4 + 2.2 + 0.0 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 14 \\ 3 & 2 \\ 5 & 8 \end{pmatrix}$$

Here A is a matrix of order 3×3 and B is a matrix of order 3×2. Hence BA does not exist as number of columns in B is not equal to the number of rows in A.

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. If $A = \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}$, find $A^2 + 3A + 5I$ where I is unit matrix of order 2.
2. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$. Find a matrix C such that $A + B = 2C$.
3. If $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$, find A^3.
4. Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \\ 5 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 5 & 9 \\ 6 & -2 & 1 \end{pmatrix}$
   (i) Write down the order of the matrices A and B.
   (ii) Write down the order of the product AB.
   (iii) Calculate AB.
   (iv) Is it possible to calculate BA?
   (v) *Is* $AB = BA$?
   (vi) Are the following possible for operation?
   $A + B$, $A - B$, $2B$ and A^2

Lesson-3: Determinant

After studying this lesson, you should be able to:

- State the concept of determinant;
- Describe the advantages of determinant;
- Express the Cramer's rule;
- Solve the system of linear equations by Cramer's Rule.

Introduction

The present lesson is devoted to a brief discussion of determinants and their more elementary properties. The determinant concept is of a particular interest in solving simultaneous equations.

Determinant

An important concept in matrix algebra is that of the determinant. If a matrix is square, the elements of the matrix may be combined to compute a real-valued number called the determinant and is denoted either by the symbol Δ, or by placing vertical lines around the elements of the matrix (like |**A**|) or simply by *det.A*. The signs of the successive terms in the expansion of determinants will be alternately positive and negative until the last term is reached.

$$\text{If, } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Determinant of A will be denoted by $\Delta = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Types of Determinants

First Order Determinant: A determinant of the first order is defined by the determinant of a 1×1 Matrix. The determinant of a 1×1 matrix is simply the value of the one element contained in the matrix.

Let, $A = [a_{11}]$ be a square matrix. Then $|A| = a_{11}$ be a determinant of first order.

Second Order Determinant: A determinant of the second order is defined by the determinant of a 2×2 Matrix.

Let, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a 2×2 matrix and the determinant of A is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

That is the value of the determinant is given by the difference of the cross products.

Third Order Determinant: A determinant of the third order is defined by the determinant of a 3×3 Matrix.

Let, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3×3 matrix and the determinant of A is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Minors and Co-factors: The method discussed earlier applies for calculating the determinant of a 2×2 or 3×3 matrix. It does not, however, apply to matrices of higher dimensions. It is required a procedure for calculating a determinant that applies to any square matrix. This procedure is termed as the method of co-factor expansion. Before discussing the method of co-factor expansion, we must define two terms minor and co-factor.

Minors

The minor of an element is defined as a determinant by omitting the row and the column containing the element. Thus, a minor is the determinant of the sub matrix formed by deleting the *i*-th row and *j*-th column of the matrix.

$$\text{If a matrix, } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then – minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

minor of $a_{12} = M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

minor of $a_{13} = M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ and so on.

Co-factors

The co-factor of an element is the co-efficient of the element in the expanded form and is equal to the corresponding minor with proper sign. Thus, a co-factor is a minor with a prescribed sign. The rules for the sign of a co-factor of any element = $(-1)^{i+j}$ $\times$ its minor, where i = number of row and j = number of column.

The co-factor of $a_{ij} = c_{ij} = (-1)^{i+j} M_{ij}$

For example, co-factor of $a_{11} = (-1)^{1+1} M_{11} = M_{11}$

co-factor of $a_{12} = (-1)^{1+2} M_{12} = -M_{12}$

Example-1:

Find the minors and co-factors of the elements at the 1st row of the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 3 & 2 & 7 \end{vmatrix}$$

Solution:

The minor of the element 1, i.e., a_{11} is $M_{11} = \begin{vmatrix} 5 & 0 \\ 2 & 7 \end{vmatrix} = 35$

The minor of the element 2, i.e., a_{12} is $M_{12} = \begin{vmatrix} 4 & 0 \\ 3 & 7 \end{vmatrix} = 28$

The minor of the element 3, i.e., a_{13} is $M_{13} = \begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} = -7$

The co-factor of 1, i.e., a_{11} is $C_{11} = (-1)^{1+1}\ .35 = 35$

The co-factor of 2, i.e., a_{12} is $C_{12} = (-1)^{1+2}\ .28 = 28$

The co-factor of 3, i.e., a_{13} is $C_{13} = (-1)^{1+3}\ (-7) = -7$

Expansion of Determinant and Use of Sarrus Diagram

$$\textbf{Let}\ |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

If the co-factor of a_{11}, a_{12} and a_{13} are A_{11}, A_{12} and A_{13} respectively, then

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

Sarrus Diagram: We can find out determinant value of a given matrix very conveniently by using Sarrus diagram. It is found by the following process:

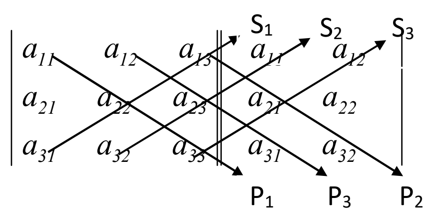
(i) Rewrite the first two columns of the matrix to the right of the original matrix.

(ii) Locate the elements on the three primary diagonals (P_1, P_2, P_3) and those on the three secondary diagonals (S_1, S_2, S_3).

(iii) Multiply the elements on each primary and each secondary diagonal.

(iv) The determinant equals the sum of the products for the three primary diagonals minus the sum of the products for the three secondary diagonals.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the determinant may be found by the following process



Thus, algebraically the determinant value is computed as

$|A| = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$

Hence expansion of the determinant of $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ **will be**

$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$

$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

Example-2:

Find the value of $\begin{vmatrix} 1 & 5 & 3 \\ 2 & 0 & 5 \\ -4 & 1 & -2 \end{vmatrix}$

Solution:

Let D = $\begin{vmatrix} 1 & 5 & 3 \\ 2 & 0 & 5 \\ -4 & 1 & -2 \end{vmatrix}$

$= 1\,(0 - 5) - 5\,(-4 + 20) + 3\,(2 - 0)$
$= (-5 - 80 + 6) = 79.$

Properties of Determinants

Certain properties hold for determinants. The following properties can be useful in computing the value of the determinant.

- If two rows or columns are interchanged in a determinant, the sign of the determinant changes but its value is unchanged.
- If rows are changed into columns and columns into rows, the determinant remains unchanged.
- If two rows or columns are identical in a determinant, it vanishes.
- If all the elements of any row or column are zero, the determinant is zero.
- If any multiple of one row or column is added to another row or column, the value of the determinant is unchanged.
- If any row or column is a multiple of another row or column, the determinant equals to zero.

Example-3:

Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution:

Applying $C'_1 = C_1 - C_2$; $C'_2 = C_2 - C_3$ we get

$$\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$$

$$= (a-b)\,(b-c)\begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix}$$

$$= (a-b)\,(b-c)\,(c-a)$$

Example-4:

Show that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

Solution:

Applying $C'_1 = C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Applying $R'_1 = R_1 - R_2$; $R'_2 = R_2 - R_{23}$

$$= 2(a+b+c) \begin{vmatrix} 0 & -(a+b+c) & 0 \\ 0 & (a+b+c) & -(a+b+c) \\ 1 & a & c+a+2b \end{vmatrix}$$

$= 2(a+b+c)^3$

Cramer's Rule and Its use in the Solution of Equations

Cramer's rule is a simple rule using determinants to express the solution of a system of linear equations for which the number of equations is equal to the number of variables. This rule states $\bar{x}_i = \frac{D_i}{D}$ where x_i is the *i*-th unknown variable in a series of equations, D is the determinant of the coefficient matrix, and D_i is the determinant of a special matrix formed from the original coefficient matrix by replacing the column of coefficients of x_i with the column vector of constants. Thus, Cramer's rule can be fruitfully applied in case $D \neq 0$.

Example-7:

Solve the following system of equations by using Cramer's Rule.

$5x - 6y + 4z = 15$

$7x + 4y - 3z = 19$

$2x + y + 6z = 46$

Solution:

Here $D = \begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix} = 419$

$D_x = \begin{vmatrix} 15 & -6 & 4 \\ 19 & 4 & -3 \\ 46 & 1 & 6 \end{vmatrix} = 1257$

$D_y = \begin{vmatrix} 5 & 15 & 4 \\ 7 & 19 & -3 \\ 2 & 46 & 6 \end{vmatrix} = 1676$

$D_z = \begin{vmatrix} 5 & -6 & 15 \\ 7 & 4 & 19 \\ 2 & 1 & 46 \end{vmatrix} = 2514$

We know from the Cramer's Rule, $\frac{x}{D_x} = \frac{y}{D_y} = \frac{z}{D_z} = \frac{1}{D}$

Hence $x = \frac{D_x}{D} = \frac{1257}{419} = 3$

$y = \frac{D_y}{D} = \frac{1676}{419} = 4$

$z = \frac{D_z}{D} = \frac{2514}{419} = 6.$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Find all the minors and co-factors of the following determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -2 & 8 & 1 \end{vmatrix}$$

2. Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

3. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

4. Find the value of $\begin{vmatrix} x+y & x & y \\ x & x+z & z \\ y & z & y+z \end{vmatrix}$

5. Solve the following system of equations by using Cramer's Rule:

   $x+5y-z=9$

   $3x-3y+2z=7$

   $2x-4y+3z=1$

6. Solve the following system of equations by using Cramer's Rule:

   $x-y+z=1$

   $x+y-2z=0$

   $2x-y-z=0$

7. Solve the equation $\begin{vmatrix} p+x & q+x & r+x \\ q+x & r+x & p+x \\ r+x & p+x & q+x \end{vmatrix} = 0$

Lesson-4: Matrix Inversion

After studying this lesson, you should be able to:

- Explain inverse matrix;
- Solve system of linear equations by inverse matrix method.

Introduction

The operation of dividing one matrix directly by another does not exist in matrix theory but equivalent of division of a unit matrix by any square matrix can be accomplished (in most cases) by a process known as inversion of matrix. The concept of inverse matrix is useful in solving simultaneous equations, input-output analysis and regression analysis.

Inverse Matrix

If A is a square matrix of order n, then a square matrix B of the same order n is said to be inverse of A if $AB = BA = I$ (unit matrix).

Methods of Matrix Inversion

There are several methods for determining the inverse of a matrix; two of these are discussed in below.

(i) Co-factor matrix method.
(ii) Gauss- Jordan Elimination method.

Working Rule for Inverse Matrix (Co-factor matrix method)

To evaluate the inverse of a square matrix A, we should follow the following steps:

(i) Evaluate $|A|$ for the matrix A and be sure that $|A| \neq 0$
(ii) Calculate the co-factors of all the elements of the matrix A.
(iii) Find the matrix of the co-factor A^C.
(iv) Then find the Adjoint of A by taking transpose of A^C such that Adj A = $(A^C)^T$.
(v) Finally divide all the elements of Adj A by $|A|$ to get the required inverse A^{-1}.

Example-1:

Find the inverse of the matrix, $A = \begin{bmatrix} 2 & 4 \\ 3 & 8 \end{bmatrix}$

Solution:

The determinant of the matrix A is, $|A| = \begin{vmatrix} 2 & 4 \\ 3 & 8 \end{vmatrix} = 4 \neq 0$

The co-factor matrix of A is, $A^C = \begin{bmatrix} 8 & -3 \\ -4 & 2 \end{bmatrix}$

The Ad joint matrix of A is, $A^j = \begin{bmatrix} 8 & -4 \\ -3 & 2 \end{bmatrix}$

Therefore, the inverse of A is,

$$A^{-1} = \frac{1}{\Delta} A^j = \frac{1}{4} \begin{bmatrix} 8 & -4 \\ -3 & 2 \end{bmatrix}$$

Example-2:

Find the inverse of the matrix, A = $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ -1 & 3 & 2 \end{bmatrix}$

Solution:

The determinant of the matrix A is, $|A| = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ -1 & 3 & 2 \end{vmatrix} = 1$

The co-factor matrix of A is, $A^C = \begin{bmatrix} 3 & -1 & 3 \\ -4 & 2 & -5 \\ -2 & 1 & -2 \end{bmatrix}$

The Adjoint matrix of A is, $A^J = \begin{bmatrix} 3 & -4 & -2 \\ -1 & 2 & 1 \\ 3 & -5 & -2 \end{bmatrix}$

Therefore, the inverse of A is,

$$A^{-1} = \frac{1}{\Delta} A^J = \frac{1}{1}\begin{bmatrix} 3 & -4 & -2 \\ -1 & 2 & 1 \\ 3 & -5 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -4 & -2 \\ -1 & 2 & 1 \\ 3 & -5 & -2 \end{bmatrix}$$

Gauss-Jordan Elimination Method

To determine the inverse of an $m \times m$ matrix 'A', following are the steps

(i) Determining the determinant value of A, whether it is non-singular or not.

(ii) Augmenting the matrix A with an $m \times m$ identity matrix, resulting in ($A \mid I$).

(iii) Performing row operations on the entire augmented matrix so as to transform 'A' into an $m \times m$ identify matrix. The resulting matrix will have the following form ($I \mid A^{-1}$) where, the A^{-1} can be read to the right of the vertical line.

Example-3:

Find the inverse of the matrix, $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$

Solution:

Augmented the matrix 'A' by 2×2 identity matrix, we get –

$$\left[\begin{array}{cc|cc} 3 & 7 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{3} & \frac{1}{3} & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \text{applying } r_1^{/} = r_1 \times \frac{1}{3}$$

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right] \text{applying, } r_2^{/} = r_2 - r_1 \times 2$$

$$\begin{bmatrix} 1 & \frac{7}{3} & | & \frac{1}{3} & 0 \\ 0 & 1 & | & -2 & 3 \end{bmatrix} \text{applying, } r_2^{/} = r_2 \times \frac{1}{3}$$

$$\begin{bmatrix} 1 & 0 & | & 5 & -7 \\ 0 & 1 & | & -2 & 3 \end{bmatrix} \text{applying, } r_1^{/} = r_1 - r_2 \times \frac{7}{3}$$

So, the inverse of 'A' is, $A^{-1} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

Solution of Linear Equations by Using Inverse of Matrix

Matrix algebra permits the concise expression of a system of linear equations. The inverse matrix can be used to solve a system of simultaneous equations. Let a system of linear equations are:

$$a_{11}x + a_{12}y + a_{13}z = k_1$$

$$a_{21}x + a_{22}y + a_{23}z = k_2$$

$$a_{31}x + a_{32}y + a_{33}z = k_3$$

It can be written in the matrix form as follows:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

$$AX = B \text{ ;where,} A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \; B = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

$$X = A^{-1}B$$

Example-4:

Use matrix inversion to solve the following system of equations

$$4x_1 + x_2 - 5x_3 = 8$$

$$-2x_1 + 3x_2 + x_3 = 12$$

$$3x_1 - x_2 + 4x_3 = 5$$

Solution:

The given system of equations can be written in the matrix form

$$\begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\text{Now } |A| = \begin{vmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{vmatrix} = 98$$

The co-factor matrix of A is $A^C = \begin{bmatrix} 13 & 11 & -7 \\ 1 & 31 & 7 \\ 16 & 6 & 14 \end{bmatrix}$

The Adjoint matrix of A is, $A_j = \begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}$

$\therefore$ The inverse of A is, $A^{-1} = \dfrac{1}{\Delta} A_j = \dfrac{1}{98} \begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}$

$$X = A^{-1}B = \frac{1}{98} \begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

$\therefore x_1 = 2, x_2 = 5, x_3 = 1.$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Find the inverse of the matrix, $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & 2 \end{bmatrix}$
2. Find the inverse of the matrix, $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$
3. Solve the following system of equations by using Gaussian method.
   $2x - 5y + 7z = 6$
   $x - 3y + 4z = 3$
   $3x - 8y + 11z = 11$
4. Use matrix inversion to solve the following system of equations:
   $x + y + z = 3$
   $x + 2y + 3z = 4$
   $x + 4y + 9z = 6$
5. Use matrix inversion to solve the following system of equations:
   $x + 2y + 3z = 6$
   $2x + 4y + z = 7$
   $3x + 2y + 9z = 14$