Functions, Limit and Continuity of a Function

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From the discussion of this unit, students will be familiar with different functions, limit and continuity of a function. The principal foci of this unit are nature of function and its classification, some important limits and continuity of a function and its applications followed by some examples.

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Lesson-1: Functions

After studying this lesson, you should be able to:

- Discuss the nature of variable and constants;
- State the functions and its classification;
- Highlights on some worked out examples related to the functions.

Introduction:

First of all we have to know some important terms, which are frequently used in this lesson. These are:

- **Constant**

  A constant is a symbol - which never changes over the sets of mathematical operation. For example, 1, 2, 3 are constants. The letter *a, b, c* --- are also considered as constants which are specially know as arbitrary constants.

- **Variables**

  A symbol capable of assuming different values is called a variable. Variable are usually denoted by the letters of the alphabet; i.e., *x, y, z*.

- **Independent Variable**

  A variable to which any value can be assigned is called an independent variable. Independent variables are the causes and the dependent variables are the effects.

- **Dependent Variable**

  A variable whose value depends on the value of the independent variable is called a dependent variable.

- **Function**

  When two variables are so related that one is dependent and another is independent, then the dependent variable is known as function of independent variable. For example, let us consider two variables x and y, which are related by the equation $y = 4x + 6$. We see that if we take $x = 1$, then we get $y = 10$; if we take $x = 0$, we get $y = 6$ and thus we see here that x is independent variable and y is dependent variable. So we may say that y is the function of x which is denoted by the symbol, $y = f(x)$. Hence we may conclude that any expression containing a variable is called a function of that variable. Thus *(i)* $ax + 10$, *(ii)* $2x^2 - 5x + 2$, *(iii)* $t^2 - 1$ and *(iv)* $e^t - 5$, where the expressions *(i)* and *(ii)* are functions of x and the expressions *(iii)* and *(iv)* are functions of t. The related variable on which the value of the function depends is also known as argument of the function.

When two variables are so related that one is dependent and another is independent, then the dependent variable is known as function of independent variable

Type of Functions

The different types of functions have been discussed as under:

a) Linear Function: A linear function represents a relationship between two variables, i.e., one dependent variable and another independent variable. Generally the functional form of the linear function is, $f(x) = ax + b$

where, $f(x)$ is the dependent variable
x is the independent variable
b is the value of the dependent variable when x is zero.
a is the coefficient of the independent variable.

The above symbol $f(x)$ is read as *"function of x"*, which represents the values of the dependent variable, and x represents values of the independent variable; $f(x)$ varies according to the rule of the function as x varies. For a linear function, the rule of the function states that *'a'* is to be multiplied by x and this product is to be added to b. This sum determines the value of the dependent variable $f(x)$.

b) Quadratic Function: The quadratic function is a second degree function which has important applications in business and economics. The general form of the quadratic function is, $f(x) = ax^2 + bx + c$

where, $f(x)$ is the dependent variable
x is the independent variable
a, b and c are the parameters of the functions.

The shape of the quadratic function is determined by the magnitude and signs of the parameters a, b and c.

c) Polynomial Functions: Linear and quadratic functions belong to the class of functions termed polynomial functions. The general form of the polynomial function is

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$

where, $a_0, a_1, a_2, a_3, ... a_n$ are parameters and n is a positive integer.

The parameters may be positive, negative or zero. The polynomial function is linear if $n=1$ and quadratic if $n=2$. This can be verified by comparing this for $n=1$ with the general form of the linear function, and $n=2$ with the general from of the quadratic function.

A polynomial function in which the largest exponent is n=3 is termed as cubic function

A polynomial function in which the largest exponent is $n=3$ is termed as cubic function. The general form of the cubic function is $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

d) Multivariate Functions: Functions in which the single dependent variable is related to more than one independent variable are termed as multivariate functions. The general form of multivariate function is, $f(x_1, x_2) = 2x_1+5x_1 x_2+6x_2$

where $f(x_1, x_2)$ is the dependent variable
x_1 is an independent variable
and x_2 is a second independent variable.

e) Exponential Functions: The exponential function is a specific function in which a constant is raised to a variable power rather than a variable raised to constant power. This function with a variable power is called the exponential function. The general form of exponential function is , $h(x) = k.a^{f(x)}$

where $'a'$ is a constant greater than zero and not equal to one and $f(x)$ is any real number function.

The domain of this function is the set of all real numbers, x, for which $f(x)$ is defined.

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The exponential function states the constant rates of growth. As the independent variable increases by a constant amount in the exponential function, the dependent variable increases or decreases by a constant percentage. Hence, the value of an investment that increases by a constant percentage each period, the sales of a company that increase at a constant rate each period, and the value of an asset that declines at a constant rate each period are examples of functional relationship that are described by the exponential functions.

f) Logarithmic Functions: The inverse of the exponential function is the logarithmic function. The general form of the logarithmic function is, $y = \log_a x$

where, y is the dependent variable
x is the independent variable
and $'a'$ is a constant, termed the base, that is greater than 0 and not equal to 1.

The logarithmic function arises when we ask the question, for what value of y is $a^y = x$.

If $a^y = x$, then $\log_a x = y$ and vice-versa. Thus, the exponential function is corresponding to the logarithmic function, $y = \log_a x$ is $a^y = x$.

Rate of Change

The rate of change of a function is the change in the value of the dependent variable with respect to the change in the value of the independent variable, i.e.,

$$\text{Rate of change} = \frac{\Delta f(x)}{\Delta x} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}}$$

If the independent variable x increases by Δx, the new value of the independent variable is $(x + \Delta x)$. For a linear function when $f(x) = ax + b$, the new value of the dependent variable for change in x is $f(x + \Delta x) = a(x + \Delta x) + b$.

To determine the amount of change in the dependent variable as the independent variable changes by Δx, the old value of the dependent variable, $f(x)$ is subtracted from the new value of the dependent variable, $f(x + \Delta x)$. That is, for the linear case,

$$\Delta f(x) = f(x + \Delta x) - f(x) = a(x + \Delta x) + b - (ax + b)$$

$$= (ax + a\Delta x + b - ax - b) = a.\Delta x.$$

Hence for the linear case, the rate of change is $\frac{\Delta f(x)}{\Delta x} = \frac{a.\Delta x}{\Delta x} = a,$ which is the slope. The rate of change can be calculated for any function, linear or non-linear, using the same formula.

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Notations for Functions

It *'x'* is a variable of a function, then it is expressed as *f(x), F(x), g(x), ...* $f_1(x), f_2(x)$ ... which are basically called *functions of x*. Similarly it may be expressed as *'the f function of x'*, *'the F function of x'* ... etc.

Again, if more than one variable (*x, y, z*) exist in a particular function, it can be expressed as *f(x, y), F(x, y, z)*. It is termed as *'the function of x and y'*, *'the F function of x, y, and z'* etc.

For example, If $f(x) = 2x^3 - 5x + 3$ and $F(x, y) = 3x^c + 5e^y - 3xy$,

then, $f(p)=2p^3 - 5p + 3$ and $F(b, d) = 3b^c + 3e^d - 3bd.$

If the interval is $a \le x \le b$, then it is called closed domain in which the values of a and b are included.

If the value of *'x'* exists between *a* and *b* then it is termed as domain or interval. If the interval is $a \le x \le b$, then it is called closed domain in which the values of *a* and *b* are included.

Again if the interval is $a < x < b$, then it is called open domain, where the mid values of a and b are included only.

Again if the interval is $a < x < b$, then it is called open domain, where the mid values of *a* and *b* are included only. The samples of functions are presented as under:

$f(x) = 3x + 5$ → It is a linear function

$f(x) = 3x^2 - 3x + 8$ → It is a quadratic function

$f(x) = 4x^3 - 9x^2 + 3x - 6$ →It is a cubic function.

The following examples illustrate the use of functions

Example-1:

If $p(q) = q^2 - r^2 + 5$ and $h(r) = q^2 - r^2 + 5$; what is (i) *p (2)* and (ii) *h (3)*?

Solution:

(a) We are given, $p(q) = q^2 - r^2 + 5$

$\therefore p(2) = (2^2 - r^2 + 5) = 9 - r^2$

(b) We are given, $h(r) = q^2 - r^2 + 5$

$\therefore h(3) = (q^2 - 3^2 + 5) = q^2 - 4$

Example-2:

Find *g(64)*, If $g(x) = \frac{x^{\frac{3}{2}}}{32} - 16x^{-\frac{1}{2}} + 2x^{\frac{1}{3}}$

Solution:

We are given, $g(x) = \frac{x^{\frac{3}{2}}}{32} - 16x^{-\frac{1}{2}} + 2x^{\frac{1}{3}}$

$$\therefore\ g(64) = \left[\frac{64^{\frac{3}{2}}}{32} - 16(64)^{-\frac{1}{2}} + 2(64)^{\frac{1}{3}}\right] = (16 - 2 + 8) = 28$$

Example-3:

Find (i) $g(a) - g(x - a)$, if $g(x) = x^2 + 10$

(ii) $f(x + a) - f(x)$; if $f(x) = x^2 - 3$

Solution:

(i) We are given $g(x) = x^2 + 10$

$$\therefore\ g(a) - g(x - a) = (a^2 + 10) - \{(x - a)^2 + 10\}$$
$$= a^2 + 10 - x^2 + 2xa - a^2 - 10$$
$$= 2xa - x^2$$

(ii) We are given, $f(x) = x^2 - 3$

$$\therefore\ f(x + a) - f(x) = (x + a)^2 - 3\ - (x^2 - 3)$$
$$= x^2 + 2xa + a^2 - 3 - x^2 + 3$$
$$= 2xa + a^2$$

Example-4:

If $f(x) = \frac{ax+b}{bx+a}$, prove that $f(x).\ f(\frac{1}{x}) = 1$

Solution:

We have $f(x) = \frac{ax+b}{bx+a}$

Replacing x by $\frac{1}{x}$ on both sides, we get

$$f(\frac{1}{x}) = \frac{a.\frac{1}{x}+b}{b.\frac{1}{x}+a} = \frac{a+bx}{b+ax}$$

$$\therefore\ f(x).f(\frac{1}{x}) = \frac{ax+b}{bx+a} \times \frac{a+bx}{b+ax} = 1 \text{ (Proved)}$$

Example-5:

Find the domain of the following function $\frac{x^2 + x + 5}{x^2 - 6x + 8}$

Solution:

Let $f(x) = \frac{x^2 + x + 5}{x^2 - 6x + 8}$

Clearly, $f(x)$ will be undefined if

$x^2 - 6x + 8 = 0$

or, $x^2 - 4x - 2x + 8 = 0$

or, $x(x - 4) - 2(x - 4) = 0$

or, $(x - 4)(x - 2) = 0$

$\therefore x = 2$ or $x = 4$

Hence, the domain of the definition of $f(x)$ is:

$-a < x < a$, but $x \neq 2$ and $x \neq 4$.

Example-6:

If $e^y - e^{-y} = 2x$, express y as an explicit function of x.

Solution:

We have $e^y - e^{-y} = 2x$ (Let $z = e^y$)

or, $z - \frac{1}{z} = 2x$

or, $z^2 - 1 = 2zx$

or, $z^2 - 2xz - 1 = 0$

$$\therefore z = \frac{2x \pm \sqrt{4x^2 - 4.1.(-1)}}{2.1}$$

$$\text{or, } e^y = \frac{2x \pm 2\sqrt{x^2 + 1}}{2}$$

or, $e^y = x \pm \sqrt{x^2 + 1}$

or, $\log_e e^y = \log_e (x \pm \sqrt{x^2+1})$

or, $y = \log_e (x \pm \sqrt{x^2+1})$, which expresses y as an explicit function of x.

Example-7:

The taxi fare is Tk.10 for 1 km or less from start and Tk.5 per km or any fraction thereof for additional distance. If the fare be Tk.y for a distance of x km, express y as a function of x.

Solution:

From the problem it is clear that

$y = 10$ when $0 < x \leq 1$

and $y = 10 + 5$ when $1 < x \leq 2$

$y = 10 + 2(5)$ when $2 < x \leq 3$

$y = 10 + 3(5)$ when $3 < x \leq 4$.

and in general,

$y = 10 + p(5)$ when $p < x \leq p+1$

when $p = 0$ or a positive integer.

Hence, the functional relation between y (in Tk.) and the distance traveled x (in km) is given by $y = 10 + p(5)$ when $p < x \leq p+1$, where $p = 0$ or a positive integer.

Example-8:

Find the range of the function $\frac{x}{1+x^2}$

Solution:

Let $y = \frac{x}{1+x^2}$

or, $x^2y + y = x$

or, $x^2y - x + y = 0$

$\therefore\ x = \frac{1+\sqrt{1-4y^2}}{2y}$

Since x is finite and real, we have

$y \neq 0$, and $1 - 4y^2 \geq 0$

or, $(1 - 2y)\ (1+2y) \geq 0$

$\therefore\ -{}^1/_2 \leq y \leq {}^1/_2$

Therefore, the required range of the given function is: $-\frac{1}{2} \leq y \leq \frac{1}{2}$ *and* $y \neq 0$

or, $-{}^1/_2 \leq y < 0$ and $0 < y \leq {}^1/_2$.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. What do you mean by constant and variable?
2. Define a function. What do you mean by domain interval and range of a function?
3. If $f(x-1) = 7x-5$, find $f(x)$ and $f(x+2)$
4. If $f(x) = x^2 - x$, then prove that $f(h+1) = f(-h)$
5. If $f(x) = \frac{1}{x^2}$ show that $f(x+h) - f(x-h) = -\frac{4xh}{(x^2-h^2)^2}$
6. If $f(x) = \log_e \frac{1+x}{1-x}$, show that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$
7. If $y = f(x) = \frac{x-3}{2x+1}$ and $z = f(y)$, express z as a function of x.
8. Find the domain of the following function $\frac{x-2}{x^2-3x+2}$
9. Find the range of the function $\frac{x^2}{1+x^2}$

Multiple Choice Questions (Find the most appropriate answer)

1. Given $f(x) = 3x - 9$; find $f(x^2 - 1)$

   (i) $x^2 - 12$ (ii) $3x^2 - 12$
   (iii) $4x^2 - 6$ (iv) $3x^2 - 10$

2. Find the domain of the functions $y = \frac{x^2}{1+x^2}$

   (i) $x \le 2$ and $x \ge 5$ (ii) $x < -6$ and $x > e$
   (iii) $x > 2$ and $x < -1$ (iv) $a < x < a$

3. Find the range of the function $y = \frac{x}{x^2-5x+9}$

   (i) $0 \le y \le 2$ (ii) $0 \le y \le 1$
   (iii) $-\frac{1}{11} \le y \le 1$ (iv) $y \le \frac{1}{2}$ and $y \ge 9/2$

4. If $\log x + \log y = 2x$, express y as an explicit function of x:

   (i) $y = \frac{bx+d}{ax+c}$ (ii) $y = \frac{c^{2x}}{x}$
   (iii) $y = \frac{e^{2x}+2}{2x}$ (iv) $y = \log_e (x \pm \sqrt{x^2-1})$

5. If $f(x) = 10x^2 - 13x + 13$, solve the equation $f(x) = 16$

   (i) 2, 5 (ii) $^2/_5, ^1/_3$ (iii) 3, 7 (iv) $^3/_2, ^{-1}/_5$

6. If $y = f(x) = \dfrac{3x-5}{2x-m}$ and $f(y) = x$, find the value of m.

(i) $m = 6$ (ii) $m = 3$ (iii) $m = 7$ (iv) $m = 8$

Lesson-2: Limit

After studying this lesson, you should be able to:

- Discuss the nature of fundamental theorems on limit;
- Apply the different methods of evaluating the limit.

Introduction

Limit determines whether the value of a function exists in the neighborhood of a point at which the function is undefined

The concept of limit is an operation, which determines whether the value of a function exists in the neighborhood of a point at which the function is undefined. It is completely new concept in mathematics and is considered to be the basis of calculus. Now-a-days, this concept has wide application in the theoretical discussion in different branches of science including mathematics and in the solution of different problems in economics. In this lesson we shall make a brief discussion on the limit of a function and the application of fundamental theorems in evaluating limit of a function.

Limit of a Function

Generally, we are concerned with what happens to the value of the dependent variable *f(x)* as the value of the independent variable *x* approaches some constant *a*. For example, the function *f* defined by $f(x) = x+2$ and notice what happens to the value of *f(x)* as the value of *x* moves closer and closer to 2.

Let us set up a table of *x* and corresponding *f(x)* values, as

x : 1.9 → 1.99 →1.999 →1.9999   2.0001 → 2.001 → 2.01 → 2.1

f(x): 3.9 → 3.99 → 3.999 → 3.9999   4.0001 → 4.001 → 4.01 → 4.1

Fundamental Theorems of Evaluating Limit of a Function

The following theorems are most useful in the evaluation of limits.

For any real number *a*, assuming that $\lim_{x\to a} f(x)$ and lim $\lim_{x\to a} g(x)$ exist:

1. For any real constant *k*, $\lim_{x\to a} k = k$

2. For any real number n, $\lim_{x\to a} x^n = a^n$

3. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ in the root is defined

4. $\lim_{x\to a} k.f(x) = k.\lim_{x\to a} f(x)$

5. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

6. $\lim_{x\to a} f(x).g(x) = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$

7. $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} f(x) / \lim_{x\to a} g(x)$; if $\lim_{x\to a} g(x) \neq 0$

8. If n is any positive integer, then

(a) $\lim\limits_{x\to\infty^+} \frac{1}{x^n} = 0$ (b) $\lim\limits_{x\to\infty^-} \frac{1}{x^n} = 0$

9. If n is any positive integer, then

$\lim\limits_{x\to 0^+} \frac{1}{x^n} = +\infty$ and $\lim\limits_{x\to 0^+} \frac{1}{x^n} = +\infty$ (if n is even) or $-\infty$ (if n is odd)

10. $\lim\limits_{x\to a} \log f(x) = \log \lim\limits_{x\to a} f(x)$

11. $\lim\limits_{x\to a} [f(x)]^n = [\lim\limits_{x\to a} f(x)]^n$

The following examples illustrate the uses of these theorems.

Example-1:

Compute $\lim\limits_{x\to 0} \frac{\sqrt{3+x}-\sqrt{3-x}}{x}$

Solution:

$$\lim_{x\to 0} \frac{\sqrt{3+x}-\sqrt{3-x}}{x} \times \frac{\sqrt{3+x}+\sqrt{3-x}}{\sqrt{3+x}+\sqrt{3-x}}$$

$$= \lim_{x\to 0} \frac{(3+x)-(3-x)}{x.[\sqrt{3+x}+\sqrt{3-x}]} = \lim_{x\to 0} \frac{2}{\sqrt{3+x}+\sqrt{3-x}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Example-2:

Evaluate $\lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h}$; where, $g(x) = 7x + 9$

Solution:

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0} \frac{[7(x+h)+9]-[7x+9]}{h}$$

$$= \lim_{h\to 0} \frac{7x+7h+9-7x-9}{h} = \lim_{h\to 0} \frac{7h}{h}$$

$= \lim\limits_{h\to 0} 7 = 7.$ The constant function 7 is continuous.

Example-3:

Evaluate $\lim\limits_{x\to 0} \frac{e^x - e^{-x}}{x}$

Solution:

$$\lim_{x\to 0}\frac{e^x - e^{-x}}{x}$$

$$= \lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+.........\right)-\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-.....\right)}{x}$$

$$= \lim_{x\to 0}\frac{2\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+.......\right)}{x}$$

$$= \lim_{x\to 0} 2\left(1+\frac{x^2}{3!}+\frac{x^4}{5!}+\frac{x^6}{7!}+.......\right) = \lim_{x\to 0} (2\times 1) = 2$$

Example-4:

Prove that $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

Solution:

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

$$= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x} = \lim_{x\to 0}\frac{Sin^2\frac{x}{2}}{\frac{x}{2}}$$

$$= \lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{\frac{x}{2}}\times\lim_{x\to 0}\sin\frac{x}{2} = (1\times 0) = 0$$

Example-5:

Find (a) $\lim_{x\to 2}(3x^2 - x + 6)$ (b) $\lim_{x\to 3}(2x^2+1)(3x-4)$

Solution:

(a) $\lim_{x\to 2}(3x^2 - x + 6)$

$= \lim_{x\to 2} 3x^2 - \lim_{x\to 2} x + \lim_{x\to 2} 6$

$= [3(2)^2 - 2 + 6] = (12 - 2 + 6) = 16$

(b) $\lim_{x\to 3}(2x^2+1)(3x-4)$

$= [\lim_{x\to 3} 2x^2 + \lim_{x\to 3} 1]\,[\lim_{x\to 3} 3x - \lim_{x\to 3} 4]$

$= [2.(3)^2 + 1].\,[3.(3) - 4]$

$= [(18+1).(5)] = (19 \times 5) = 95$

Example-6:

$$\lim_{x\to 2} \frac{x^2-4}{x-2}$$

Solution:

Here Substituting $x = 2$, we get $\frac{0}{0}$ which does not exist.

Hence by rationalizing, $\lim_{x\to 2} \frac{x^2-4}{x-2}$

$$= \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)} \quad [\text{as } x \neq 2;\ \therefore\ x-2 \neq 0]$$

$$= \lim_{x\to 2} (x+2) = (2+2) = 4$$

$$\therefore \lim_{x\to 2} \frac{x^2-4}{x-2} = 4.$$

Some Important Limits

The following formulae are also used for evaluating the limit of a function.

(1) $\lim_{x\to 0} \frac{x^n-a^n}{x-a} = n.a^{n-1}$

(2) $\lim_{n\to 0} (1+n)^{\frac{1}{n}} = \lim_{n\to \infty} \left(1+\frac{1}{n}\right)^n = e.$

(3) $\lim_{n\to 0} \frac{(1+x)^n-1}{x} = n$

(4) $\lim_{n\to 0} \frac{\sin x}{x} = 1$

Example-7:

Find (a) $\lim_{x\to 1} \left[2x^2(x+\sqrt{x})+3x^{\frac{1}{3}}-\frac{14}{x}\right]$ (b) $\lim_{x\to \infty} \frac{8x^2+16x+3}{2x^3-x+3}$

Solution:

(a) $\lim_{x\to 1} \left[2x^2(x+\sqrt{x})+3x^{\frac{1}{3}}-\frac{14}{x}\right]$

$$= (2\lim_{x\to 1} x^2).(\lim_{x\to 1} x + \lim_{x\to 1} \sqrt{x}) + 3\lim_{x\to 1} x^{1/3} - 14\lim_{x\to 1} \frac{1}{x}$$

$$= 2\left(1^2\right)\left(1+\sqrt{1}\right) + 3\left(1^{1/3}\right) - 14\left(\frac{1}{1}\right) = -7.$$

(b) $\lim_{x \to \infty} \frac{8x^2 + 16x + 3}{2x^3 - x + 3}$

$$= \lim_{x \to \infty} \frac{\frac{8x^2 + 16x + 3}{x^3}}{\frac{2x^3 - x + 3}{x^3}}$$

$$= \lim_{x \to \infty} \frac{\frac{8}{x} + \frac{16}{x^2} + \frac{3}{x^3}}{2 - \frac{1}{x^2} + \frac{3}{x^3}}$$

$$= \frac{0}{2} \left[where,\ \lim_{x \to \infty} \frac{1}{x} = 0 \right]$$

$= 0.$

Example-8:

Show that, $\lim_{x \to 2} \left(x^2 - 3x + 5\right) = 3$

Solution:

We have $\lim_{x \to 2} \left(x^2 - 3x + 5\right) = 3$

$= \lim_{x \to 2} x^2 - \lim_{x \to 2} 3x + \lim_{x \to 2} 5$

$= \lim_{x \to 2} x.\ \lim_{x \to 2} x - 3 \lim_{x \to 2} x + 5$

= (2 × 2 – 3 × 2 + 5) = (4 – 6 + 5) = 3 (Proved)

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Give the definition of limits of a function.

2. Mention the fundamental theorems of evaluating a function.

3. Find $\lim_{x\to 2} (3x^2 + 2)$

4. If $f(x) = \frac{1}{x}$, find $\lim_{x\to 0} f(x)$

5. Evaluate $\lim_{x\to 1} \frac{x^2 - 3x + 2}{x - 1}$

6. Evaluate $\lim_{x\to \propto} \frac{2x + 3}{x + 1}$

7. Evaluate $\lim_{x\to 0} \left\{\frac{\sqrt{1+x} - \sqrt{1-x}}{x}\right\}$

8. Prove that $\lim_{x\to 4} log\left(2x^{\frac{3}{2}} - 3x^{\frac{1}{2}} - 1\right) = 2 \log 3.$

9. Evaluate $\lim_{x\to -1} \frac{2x^2 - x - 3}{x^2 - 2x - 3}$

10. Find the value of $\lim_{x\to 0} \frac{\sqrt{1+2x} - \sqrt{1-3x}}{x}$

Multiple Choice Questions (Tick the most appropriate answer)

1. Find the value of $\lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$

    i) $\frac{1}{\sqrt{a}}$ ii) $\frac{1}{2\sqrt{a}}$ iii) $2\sqrt{a}$ iv) $\sqrt{a}$

2. Evaluate $\lim_{x\to 2} \frac{x^2 - 5x + 6}{x^3 - 5x + 2}$

    i) $\frac{1}{5}$ ii) $\frac{1}{-3}$ iii) $\frac{1}{-7}$ iv) -7

3. Evaluate $\lim_{x\to 0} \frac{\sqrt{1-x^3} - \sqrt{1-x}}{\sqrt{1+x^2} - \sqrt{1+x}}$

    i) -1 *ii)* -6 *iii)* 1 *iv)* $\frac{-1}{2}$

4. Evaluate $\lim_{x \to -1} \frac{e^{\log x} - 1}{e^{x-1} - 1}$

i) $\frac{5}{3}$ ii) $\frac{2}{7}$ iii) 1 iv) 2

5. If $g(x) = -\sqrt{25 - x^2}$, find the value of $\lim_{x \to 1} \frac{g(x) - g(1)}{x - 1}$

i) $\frac{1}{2}$ ii) $\frac{1}{2\sqrt{6}}$ iii) $\frac{1}{\sqrt{6}}$ iv) $\frac{1}{2}$

6. If $\lim_{x \to 2} \frac{ax^2 - b}{x - 2} = 4$, find the values of a and b.

i) $a = 2, b = 3$ ii) $a=1, b=3$ iii) $a=1, b=4$ iv) $a=6, b=4$.

Lesson-3: Continuity

After studying this lesson, you should be able to:

- Discuss the nature of continuity of a function;
- Apply the conditions for continuity of a function.

Nature of Continuity

A function $f(x)$ is said to be continuous in an open or closed interval if it is continuous at all points in the interval. *For example*, the function $f(x) = x^2$ is continuous in the closed interval $-4 \leq x \leq 3$ when it is continuous at every point in the interval.

A function f(x) is said to be continuous in an open or closed interval if it is continuous at all

If the function $f(x)$ is not continuous at $x = a$, we say that the function $f(x)$ is discontinuous at $x = a$ and the point $x = a$ is called a point of discontinuity of the function. The function $f(x)$ is said to be discontinuous at $x = a$ if,

(i) $f(a)$ is undefined i.e. $f(x)$ does not possess a definite finite value at $x = a$

or, (ii) $\lim_{x \to a} f(x)$ does not exist

or, (iii) $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} f(x) \neq f(a)$

Continuity of a Function

The important concept of continuity of a function is developed from the theory of limit. A function 'f' is continuous at $x = a$ if and only if all of the following conditions apply to f at a.

1) $f(a)$ is defined, i.e., the domain of f includes $x = a$;

2) $\lim_{x \to a} f(x)$ exists;

3) $\lim_{x \to a} f(x) = f(a)$, whether x approaches to a from the left or from the right.

Continuity at an Interval

If a and b are real numbers and $a < b$, then the set $\{x \mid a < x < b\}$ is called an open interval and is denoted by $[a, b]$. The set $\{x \mid a \leq x \leq b\}$ is called a closed interval and denoted by (a, b). The half-open interval $\{x \mid a \leq x < b\}$ is symbolized $(a, b]$ whereas the half-closed interval $\{x \mid a < x \leq b\}$ is symbolized $[a, b)$. In each case a and b are the endpoints of the interval, and any x value such that $a < x < b$ is interior point.

A function f is continuous on an open interval if it is continuous at each number in that interval.

A function f is continuous on an open interval if it is continuous at each number in that

A function f is continuous on a closed interval (a, b) provided the following conditions are satisfied:

1. f is continuous over the open interval (a, b)
2. $f(x) \to f(a)$ as $x \to a$ from within (a, b)
3. $f(x) \to f(b)$ as $x \to b$ from within (a, b)

The following examples illustrate the requirements/conditions for continuity of a function.

Example-1:

Show that $f(x) = \dfrac{x^2 - 4}{x - 2}$ is not continuous $x = 2$ but continuous at $x = 3$.

Solution:

The conditions to be satisfied by a function before we can say that it is continuous at a particular point say $x = a$ are: $f(a)$, $\lim\limits_{x \to a-} f(x)$ and $\lim\limits_{x \to a+} f(x)$ should have definite and finite values and these are all equal.

Let us examine whether these conditions are satisfied by $f(x) = \dfrac{x^2 - 4}{x - 2}$ for $x = 2$.

Here $x = 2$, therefore we have

(i) $f(2) = \dfrac{2^2 - 4}{2 - 2} = \dfrac{0}{0}$, which is undefined.

Again by the method of finding the left hand and right hand side limits, we have

(ii) $$\lim_{x \to 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^-} \frac{(x + 2)(x - 2)}{(x - 2)}$$
$$= \lim_{x \to 2^-} (x + 2)$$
$$= \lim_{h \to 0} (2 - h + 2) = 4$$

$\therefore$ L.H.S. limit = 4.

Again, $$\lim_{x \to 2^+} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^+} \frac{(x + 2)(x - 2)}{(x - 2)}$$
$$= \lim_{x \to 2^+} (x + 2)$$
$$= \lim_{h \to 0} (2 + h + 2) = 4$$

$\therefore$ R.H.S. limit = 4.

Here, $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) \neq f(2)$

$\therefore f(x) = \dfrac{x^2 - 4}{x - 2}$ is not continuous at $x = 2$.

Now, for $x = 3$,

i) $f(3) = \dfrac{3^2 - 4}{3 - 2} = 5$ and

ii) $\lim\limits_{x\to 3^-} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 3^-} \dfrac{(x+2)(x-2)}{(x-2)}$

$= \lim\limits_{h\to 0} (3-h+2) = 5$

$\therefore$ L.H.S. limit = 5.

Again, $\lim\limits_{x\to 3^+} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 3^+} \dfrac{(x+2)(x-2)}{(x-2)}$

$= \lim\limits_{h\to 0} (3+h+2) = 5$

$\therefore$ R.H.S. limit = 5.

Here, $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) \neq f(3)$

$\therefore f(x) = \dfrac{x^2-4}{x-2}$ is continuous at $x = 3$.

Example-2:

Show that $f(x) = 3x^2 + 2x - 1$ is continuous at $x = 2$. Also prove that $f(x)$ is continuous for all values of x.

Solution:

The condition is to be satisfied by a function if we can say that it is continuous at a particular point say $x = a$, where $\lim\limits_{x\to a^-} f(x) = f(a)$ $= \lim\limits_{x\to a^+} f(x)$

Let us examine whether these conditions are satisfied by $f(x) = 3x^2 + 2x - 1$ for $x = 2$.

Here $a = 2$, therefore, we have (i) $f(2) = (3.\ 2^2 + 2.2 - 1) = 15$.

Again by the method of finding the left and right hand side limit, we have

(ii) $\lim\limits_{x\to 2^-} (3x^2 + 2x - 1) = \lim\limits_{h\to 0} \{(3(2-h)^2 + 2(2-h) - 1\} = 15$

$\therefore$ L.H.S. limit = 15.

Again (iii) $\lim\limits_{x\to 2^+} (3x^2 + 2x - 1) = \lim\limits_{h\to 0} \{(3(2+h)^2 + 2(2+h) - 1\} = 15$

$\therefore$ R.H.S. limit = 15.

We find the values of the function at $x = 2$, the LHS limit and RHS limit and all of these exist and finite and equal. Thus the $f(x) = 3x^2 + 2x - 1$ is continuous at $x = 2$.

We shall show further that $f(x) = 3x^2 + 2x - 1$ is continuous for all values of x.

Let $x = k$ be any value of x arbitrarily selected and find out whether given function is continuous at $x = k$

Here $a = k$, therefore $f(k) = 3k^2 + 2k - 1$ (finite number) .... (1)

Also, $\lim\limits_{x \to k^-} (3x^2 + 2x - 1) = \lim\limits_{h \to 0} \{(3(k-h)^2 + 2(k-h) - 1\}$

$$= \lim_{h \to 0} (3k^2 - 6kh + 3h^2 - 2k + 2h - 1)$$

$= (3k^2 + 2k - 1)$ ....(2)

Similarly we find that, $\lim\limits_{x \to k^+} (3x^2 + 2x - 1) = 3k^2 + 2k - 1$ .... (3)

From (1), (2) and (3) we deduce that the given function is continuous at $x = k$.

Since k is any arbitrary value of x, therefore, $f(x)$ is continuous for all values of x.

Example-3:

Find the points of discontinuity of the function $\dfrac{x^2 - 3x - 4}{x^3 - 2x^2 - 5x + 6}$

Solution:

Let $f(x) = \dfrac{x^2 - 3x - 4}{x^3 - 2x^2 - 5x + 6}$

We know that if a function is undefined at $x = a$, then $x = a$ is a point of discontinuity of the function. Therefore, the points of discontinuity of $f(x)$ are the values of x at which $f(x)$ becomes undefined. The values of x for which $f(x)$ is undefined are the roots of the equation.

$x^3 - 2x^2 - 5x + 6 = 0$

or, $x^2(x - 1) - x(x - 1) - 6(x - 1) = 0$

or, $(x - 1)(x^2 - x - 6) = 0$

or, $(x - 1)(x^2 - 3x + 2x - 6) = 0$

or, $(x - 1)[x(x - 3) + 2(x - 3)] = 0$

or, $(x - 1)(x + 2)(x - 3) = 0$

$\therefore x = 1$ or $x = -2$ or $x = 3$

Hence, the points of discontinuity of $f(x)$ are : $x = 1$, $x = 3$ and $x = -2$.

Example-4:

The function $f(x) = \dfrac{2x^2 - 8}{x - 2}$ is undefined at $x = 2$. What value must be assigned to $f(2)$, if $f(x)$ is to be continuous at $x = 2$.

Solution:

We have, $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \dfrac{2(x^2 - 4)}{(x - 2)}$

$$= \lim_{x \to 2^+} \frac{2(x + 2)(x - 2)}{(x - 2)}$$

$$= \lim_{h \to 0} [2(2 + h + 2)] = 2(2+2) = 8.$$

Similarly, $\lim_{x \to 2^-} f(x) = 8$

$\therefore \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = 8$

Therefore, $\lim_{x \to 2} f(x) = 8$

Now, the function $f(x)$ will be continuous at $x = 2$ if $\lim_{x \to 2} f(x) = f(2)$; i.e., if $f(2) = 8$.

Hence the required assigned value of $f(2)$ is 8.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Define the continuity of $f(x)$ at $x = a$. When is the function said to be continuous in the closed interval $a \leq x \leq b$?
2. Define the discontinuity of $f(x)$ at $x = a$.
3. Indicate the points of discontinuity of the function: $\frac{2x^2+6x-5}{12x^2+x-20}$.
4. The function $f(x) = \frac{x^3-8}{x^2-4}$ is undefined at $x = 2$. Redefine the function so as to make it continuous at $x = 2$.
5. It $f(x) = \frac{x^2-9}{x-3}$ when $x \neq 3$; state the value of $f(3)$ so that $f(x)$ is continuous at $x = 3$.