C7: Quantitative Techniques

Lecturer Manual

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About this Lecturer Manual

The Lecturer Manual for the course C7: Quantitative Techniques contains the answers to the teacher-marked activities ONLY in the Course Manual. Some of the answers have supplementary explanations. The Lecturer Manual does not include those activities for which solutions are provided in the Course Manual.

There are ample references and extra resources mentioned in the Course Manual, which the lecturer can use for additional examples and exercises.

Summative assessments may be used as the bases for examinations. An example of an examination is given at the end of this manual.

Course outcomes

Upon completion of C7: Quantitative Techniques the student will be able to:

Outcomes

- **demonstrate** knowledge and understanding of the basic mathematical concepts applied in decision making and management;
- **analyse** data with the aid of statistical tools and basic probability theory;
- **use** electronic spreadsheets as a decision-making tool;
- **explain** the philosophy and methods of modelling;
- **apply** decision-making techniques such as linear programming, time series analysis, forecasting, regression and decision trees; and
- **reflect on** and **compare** risk management models and new theories in business and economics.

Timeframe

How long?

The expected duration of this course is 20 weeks. Since the course consists of 20 units, students will need to master approximately one unit per week. However, units vary in length and degree of difficulty.

You will have to provide some guidance to students. You may, of course, decide to make certain units optional, or leave them out, depending on the needs of your particular group of students.

The formal number of hours for the course is 120 notional hours.

Please remind students that Quantitative Techniques is a challenging course that needs hours of practice and reflection to master.

Assessment schedule

Assessments

Date:	XX/XX/XXXX
Value:	10%
Format:	XXX
Units covered:	1 – 4 (Module 1)
Date:	XX/XX/XXXX
Value:	10%
Format:	XXX
Units covered:	5 – 8 (Module 2)
Date:	XX/XX/XXXX
Value:	10%
Format:	XXX
Units covered:	9 – 12 (Module 3)
Date:	XX/XX/XXXX
Value:	10%
Format:	XXX
Units covered:	13 – 16 (Module 4)
Date:	XX/XX/XXXX
Value:	10%
Format:	XXX
Units covered:	17 – 20 (Module 5)

Final examination

Assignments

There is one final exam that you must complete for this course. This is a written closed-book exam, with the use of a pocket calculator permitted. Access to Excel is not available during the exam.

Assignment 1

Due date:	XX/XX/XXXX
Value:	50%
Format:	XXX
Modules covered:	Modules 1–5

Module 1

Mathematical tools

Introduction

This module attempts to summarise basic concepts in mathematics that underlie decision-making tools. The backgrounds and needs of students may differ vastly, which has made this a difficult module to construct.

The idea is not to make students experts in abstract mathematics, but to enable them to participate efficiently in a management environment that is becoming more knowledge-driven and quantified each year.

Upon completion of this module the student will be able to:

Outcomes

- **use** and **apply** basic mathematical operations;
- **understand** sequences and series of numbers (with application to compound interest rates, present value, future value and internal rate of return);
- **solve** linear equations and simultaneous linear equations (with application to supply, demand, break-even and budget constraints);
- **understand** and **apply** non-linear functions; and
- **understand** the roles of differentiation and integration in modelling (with application to optimisation, elasticity, surplus, growth rate and marginal cost).

Unit 1

Basic operations with numbers

Outcomes

Upon completion of this unit the student will be able to:

- **explain** the different number systems;
- **apply** various arithmetic operations to numbers (such as percentages and ratios);
- **explain** the rounding-off procedure;
- **identify** arithmetic and geometric sequences; and
- **apply** sequences and series of numbers to business situations.

Number systems and operations

Rounding off numbers

It is important to get students to realise the importance of the rounding-off phenomenon.

- Calculators and computer software round off numbers to a certain number of decimals. These differ from calculator to calculator, with the result that answers to questions can differ slightly.
- To ensure accuracy when doing calculations, as many decimal places as is reasonable should be kept up to the final answer. At that stage, a decision has to be made as to the required accuracy of the answer and this will depend on the context.
- Monetary amounts have two decimal places and quantities such as number of items should be whole numbers. It is not sensible to talk about 12.842 books or GBP 12,000.7281436. The rounded figures should be 13 books and GBP 12,000.73

Case study: EduBooks

Consider the following financial statements for EduBooks, which employs 40 people. Figures are given in pounds (millions).

Then do Activity 1.3 at the end of the statements.

Balance Sheet for EduBooks			
December 2009			
Assets		**Liabilities and Shareholders' Equity**	
Current assets		Current liabilities	
Cash	10.355	Accounts payable	18.900
Inventories	22.561	Income taxes	1.250
Total current assets	32.916	Total current liabilities	20.150
Property and equipment	55.765	Long-term debt	15.200
Less depreciation	12. 879		
Property and equipment, net	42.886	Total shareholders' equity	40.452
Total assets	75.802	Total liabilities and shareholders' equity	75.802

The balance sheet formula is: Assets = Liabilities + Shareholders' Equity

Income Statement for EduBooks		
For years ending 2005 and 2006		
	2005	2006
Net sales	*	*
Cost of sales	(2.900)	(3.485)
Gross income	15.255	18.990
Operating expenses	(2.800)	*
Income Statement for EduBooks		
For years ending 2005 and 2006		
Operating income	12.455	
Interest expense	(1.006)	(1.006)
Net profit before tax	*	*
Tax	*	*
Net income	*	*

Activity 1.3

Activity
Calculate for EduBooks

What will you do?

Calculate the following for the company:

1. Complete the income statement by filling in all cells marked *. (Assume a constant tax rate over 2005–2006 of 22% on net profit. For 2006, assume a 10% increase in operating expenses.)
2. Investigate the company's liquidity by calculating the current ratio from the balance sheet.
3. What percentage of total current assets is cash?
4. What percentage of total liabilities does shareholder's equity form?
5. What is the ratio of long-term debt to equity?
6. Measure the personnel productivity of the company by calculating the revenue per employee.
7. What is the increase or decrease in net income from 2005 to 2006?
8. If the discount rate is 7.3% p.a. for 2009, what would the present value of the total assets have been on 1 January 2009

Solutions

1.

Income Statement for EduBooks		
For years ending 2005 and 2006		
	2005	2006
Net sales	18.155	22.475
Cost of sales	(2.900)	(3.485)
Gross income	15.255	18.990
Operating expense	(2.800)	(3.08)
Operating income	12.455	15.91
Interest expense	(1.006)	(1.006)
Net profit before tax	11.449	14.904
Tax	2.519	3.279
Net income	8.93	11.625

2. Current ratio is an indication of a company's ability to meet short-term debt obligations. It is the ratio of current assets to current liabilities. The higher the ratio, the more liquid the company is. If the current assets of a company are more than twice the current liabilities, then that company is considered to have good short-term

financial strength. In this case, Current ratio = 32.916/20.150 = 1.634. The company is doing reasonably well.

3. Percentage cash = 10.355/32.916 = 0.3146 (rounded off) = 31.46%
4. Total liabilities: 75.802; Shareholder equity: 40.452

   Percentage = 40.452 / 75.802 = 0.5337 (rounded off) = 53.37%
5. Ratio: 15.2 : 40.452 = 15.2 / 40.452 = 0.3758
6. Year 2005 Revenue per employee = Net sales / number of employees
   = 18.155 / 40 = 0.453875 million
   = 453,875 pounds

   Year 2006 Revenue per employee = Net sales / number of employees
   = 22.475 / 40 = 0.0.561875 million
   = 561,875 pounds

   These are excellent figures and show increasing productivity.
7. Change in net income: 11.625 – 8.93 = 2.695 million pounds
8. Present value of total assets on 1 January 2009:

   75.802 / (1 + 0.073) = 70.645 million pounds

Sequences and series of numbers

Trends and limits for sequences

Trends and limits play an important role in analysis and forecasting. The lecturer may want to expand on this. Some students may have done calculus and will know the meaning of lim S_n as $n \to \infty$.

Series

Sums of series play a role in determining present values, internal rates of return, annuities, and so on.

Note it!

Formulas for sums of finite arithmetic and geometric series

Arithmetic sequence: $S_n = \frac{n}{2}[2T_1 + (n-1)d]$

Geometric series: $S_n = \frac{T_1(1-r^n)}{1-r} = \frac{T_1(r^n-1)}{r-1}$

Activity 1.6

Activity

Identify sequences and discuss trends

What will you do?

1. Write down the first six terms of each sequence from the formula for these general terms.

   Discuss trends, if any, in the numbers.

   a) $T_n = 2n - 5$

   b) $T_n = (2n)^{-1}$

   c) $T_n = 3T_{n-1} + 2$, with $T_1 = 1$

2. Identify the sequences, where possible, and give the expression for the general term in each of these cases.

   Discuss any trends in the numbers.

   a) 5, 8, 11, 14…

   b) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16} \ldots$

   c) 3, 9, 27, 81…

3. Sales data has been collected over 12 months and the pattern is forecast to continue into the future. (The numbers are in thousands.)

   Sales data:

   1.5, 3.5, 5.5, 7.5, 9.5, 11.5, 13.5, 15.5, 17.5, 19.5, 21.5, 23.5

   a) Determine a formula to describe the sales in a general month.

   b) Use it to predict the sales in month 48.

   c) Find the total of all sales after 60 months.

Solutions

1. a) −3, −1, 1, 3, 5, 7

   b) $\frac{1}{2}; \frac{1}{4}; \frac{1}{6}; \frac{1}{8}; \frac{1}{10}; \frac{1}{12}$

   c) 1, 5, 17, 53, 161, 485

2. a) Arithmetic sequence with unbounded increasing tendency (limit = ∞).

   $T_n = 3n + 2$, $n = 1, 2, 3...$

   b) Sequence (neither arithmetic nor geometric) with bounded decreasing tendency (limit = 0). $T_n = \frac{1}{n^2}$, $n = 1, 2, 3...$

   c) Geometric sequence with unbounded increasing tendency (limit = ∞). $T_n = 3^n$, $n = 1, 2, 3...$

3. Let S_n denote sales in month n. Then $S_n = (2n - 1) + 0.5$

Sales in month 48 = S_{48} = (2 × 48 – 1) + 0.5 = 95.5 thousand = 95,500

Applications

Activity 1.9

Activity

Vary the discount rate

What will you do?

Using the information from Activity 1.8, do a sensitivity analysis by varying the discount rate to see whether the projects remain profitable under changed economic conditions.

Discuss the results.

Solutions

Let us assume that the inflation rate has increased and the required rate of return (the discount rate) increases to 10% p.a.

Project A: $NPV = -473.34$. Project A is no longer profitable.

Project B: $NPV = 157.78$. Project B is still profitable.

At a discount rate of 12% for Project B: $NPV = -815.87$. Project B is no longer profitable.

If the discount rate decreases below 8% p.a., each project will become more profitable.

Analysis of the profitability index PI will lead to the same conclusions.

Case study: Decision on project investment

A company in Jamaica has 950,000 Jamaican dollars (JMD 950,000) to invest for four years. There are three mutually exclusive investment possibilities: A, B and C.

Investment A guarantees these cash inflows at the end of each of the four years:

Year	Investment A Cash inflows in JMD
1	300,000
2	400,000
3	500,000
4	600,000

Figure 1 Cash inflows for Investment A

Investment B is a four-year fixed investment which promises a simple interest rate of 12% p.a. The total value of the investment is paid out at the end of year four and there are no annual cash inflows at the end of years one to three.

Investment C is a treasury par bond with face value JMD 950,000, which gives the investor an annual cash inflow (coupon) of 6% p.a. The annual coupon is paid out at the end of each of the four years, and the initial investment or face value is paid back at the end of year four, together with the final coupon payment.

Activity 1.10

Activity

Decision on project investment

What will you do?

Analyse the possibilities from the case study and make a recommendation about which project the company should invest in.

Follow steps 1 to 5 to come to your decision.

1. Draw up a table for annual cash inflows for investments A, B and C.
2. Calculate the required rate of return (discount rate) for the three investments, assuming a tax rate of 30% p.a. on all investments and a return rate of 10% p.a. before tax.
3. Calculate the *NPV* and *PI* for the three investments.
4. Do a sensitivity analysis on the basis of possible future interest rate changes.
5. Write a report explaining your calculations and investment recommendation.

Solutions

1. At the end of year 0 (beginning of year 1), there is a cash outflow of 950,000 JMD. This can be expressed as an inflow of −950,000 and included in the table.

Year	Investment A Cash inflows in JMD	Investment B Cash inflows in JMD	Investment C Cash inflows in JMD
1	300,000	0	57,000
2	400,000	0	57,000
3	500,000	0	57,000
4	600,000	1,406,000	1,007,000

   Investment C is a par bond, which means the price of the bond (initial value of investment) equals the face value of 950,000. The annual coupon is 0.06 × 950,000 = 57,000. At the end of year four, the face value is returned.

2. Rate of return after tax = 0.10 (1 – 0.30) = 0.07 = 7% (see "Discussion"). This is now used as discount rate for all investments.
3. Investment A: $NPV = 1{,}495{,}635.39 - 950{,}000 = 545{,}635.39$ (> 0)

   $PI = 1.574$ (> 1)

   Investment B: $NPV = 1{,}072{,}630.67 - 950{,}000 = 122{,}630.67$ (> 0)

   $PI = 1.129$ (>1)

Investment C: $NPV = 917{,}821.49 - 950{,}000 = -32{,}178.51$ (< 0)

$PI = 0.966$ (< 1)

Assuming all investment risks are equal and practically zero, Investment A is clearly the best. Investment C is not profitable since the coupon rate of 6% is lower than the required rate of 7%.

4. Various values for the discount rate can be applied. Because *NPV* and discount rates are inversely proportional, we always find *NPV* increasing with decreasing discount rates, and vice versa.
5. Students should present their analysis in the form of a report.

Activity 1.12

Activity

Calculate annuities

What will you do?

1. You pay a fixed amount of \$50 per month at the end of each month for the next 10 years. The compound interest rate is 4% p.a. How much money will you have saved after 10 years?
2. What is the *PV* of annual payments of CAD 4,000 over five years at a rate of 3.4% p.a.?
3. A company in Pakistan wants to accumulate USD 10,000 over three years at an interest rate of 4% p.a. by depositing a fixed amount at the end of every month. Assume the exchange rate will stay fixed at USD 1 = PKR 80 (Pakistani rupees). What should the monthly amount be in PKR?
4. A loan of INR 50,000 must be amortised by paying equal monthly amounts of INR 2,000. The interest rate is 2.5% p.a. How long will it take to pay off the loan?

Solutions

1. The future value is $FV = R\dfrac{(1+i)^n - 1}{i}$

   The annual rate is converted to a monthly rate: $i = 0.04/12 = 0.0033$ and the period is $10 \times 12 = 120$ months.

   $$FV = 50\,\frac{(1+0.0033)^{120} - 1}{0.0033} = 7{,}346.98 \text{ dollars}$$

2. $$PV = R\frac{(1+i)^n - 1}{i(1+i)^n} = 4000\,\frac{(1+0.034)^5 - 1}{0.034(1+0.034)^5}$$

   $$= 18{,}111.47 \text{ CAD}$$

3. Calculate monthly amount *R* in USD. Values must be monthly values.

$$R = \frac{FV \times i}{(1+i)^n - 1}$$

$$= \frac{10000 \times 0.0033}{(1.0033)^{36} - 1} = 262.06 \text{ USD}$$

In PKR: $R = 262.06 \times 80 = 2{,}0964.96$

4. $$n = \frac{\ln R - \ln(R - iPV)}{\ln(1+i)}$$

$$= \frac{\ln 2000 - \ln(2000 - (0.025/12)50000))}{\ln(1 + 0.025/12)}$$

= 25.70 months = 2 years and 1.7 month

Unit 2

Linear equations and applications

Outcomes

Upon completion of this unit the students will be able to:

- **identify** linear relations;
- **interpret** the slope (gradient) and intercepts;
- **draw** graphs of linear functions;
- **solve** linear equations and inequalities;
- **solve** simultaneous linear equations; and
- **apply** linear equations to management situations.

Linear equations and straight-line graphs

These topics are very important for applications in supply and demand, budgeting, linear programming, linear regression, expected values, and so on. Although real-life situations probably mostly involve non-linear relations, such relations are often approximated by linear expressions.

Activity 1.15

Activity

Draw graphs

What will you do?

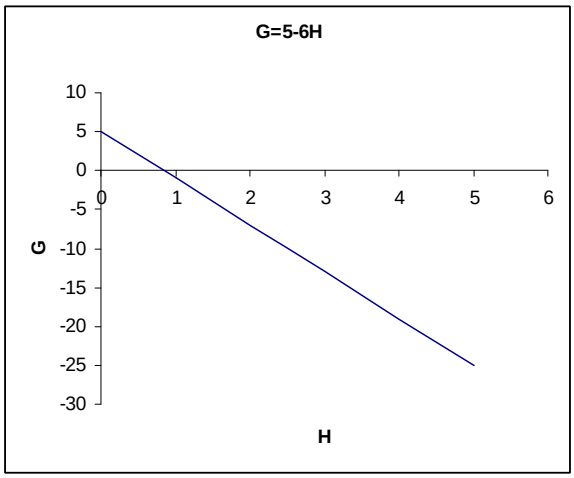
1. Draw the graph of the equation $G = 5 - 6H$ under the constraint $H \geq 0$.
2. Determine the slope and all intercepts of the line $y = 4x - 6$.
3. Draw the graph of $y = x + 3$.
4. Draw the graph and give the equation for the linear relationship between quantities x and y if you are given data points $(-2, -1)$ and $(3, 5)$. Take y as the dependent variable.
5. You are given the data points $(-1, 3)$, $(3, 6)$ and $(5, 9)$. Without plotting points, determine whether there is a linear relation between the points. Now plot the points and substantiate your answer. Discuss.
6. Write the following equations in standard form ($y = mx + c$):
   a) $4x + y - 9 = 0$
   b) $3y + 12x = 8$
7. Solve each of these inequalities by representing it as a shaded region in the (x, y)-plane:

a) $x - 2y \leq 3$

b) $3y + x + 1 \leq y - 6$

Solutions

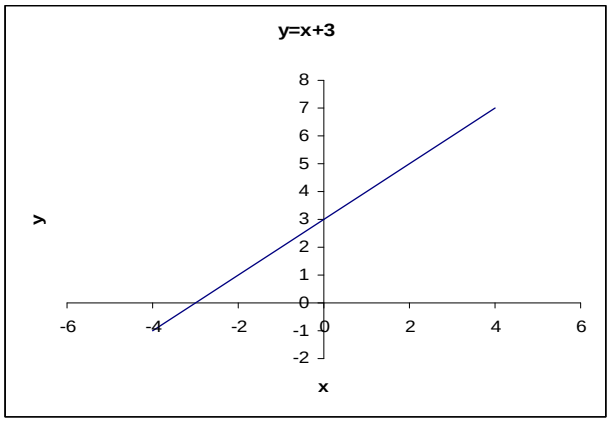
1.



2. Slope = 4. Intercept on x-axis is where $y = 0$: $4x - 6 = 0 \Rightarrow x = 1.5$

   Intercept on y-axis is where $x = 0$, $y = -6$

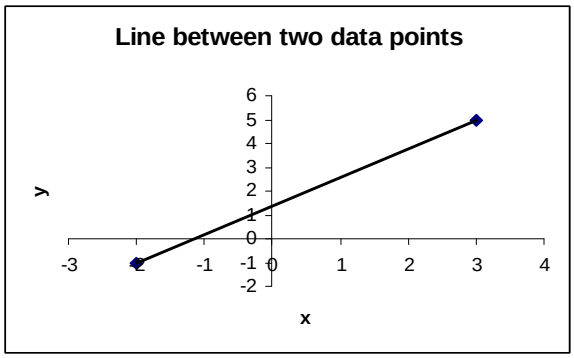
3.



   The graph actually extends indefinitely in both directions. As $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$.

4. The data points are at the extremes of the line. The line can be extended. The slope is: $m = \dfrac{5-(-1)}{3-(-2)} = 1.2$ and therefore $y = 1.2x + c$. Substitute either point, e.g. $5 = 1.2(3) + c$ gives $c = 1.4$

   Equations: $y = 1.2x + 1.4$



5. A single straight line can be drawn through any two given points in the plane. This is not necessarily true of any three given points. Determine the slope m_1 between (−1, 3) and (3, 6) and slope m_2 between (3, 6) and (5, 9).If the slopes are the same, the points lie on the same straight line and there is a linear relation. Now:

$$m_1 = \frac{6-3}{3-(-1)} = 0.75$$

$$m_2 = \frac{9-6}{5-3} = 1.5$$

There is no linear relation. The graph will be piecewise linear.

6. a) $y = -4x + 9$

   b) $y = -4x + 8/3$

   The equations therefore represent parallel lines.

7. a) Write in the form $y \geq x/2 - 3/2$. Draw line $y = x/2 - 1.5$ with y-intercept at −1.5 and slope ½. Test for (0, 0) to satisfy inequality:
   $0 \geq 0/2 - 3/2$ is true. The region lies above the line and includes origin (0, 0).

   b) Write in the form $y \leq -x/2 - 7/2$. Draw line $y = -x/2 - 3.5$ with y-intercept at −3.5 and slope −½. Test for (0, 0) to satisfy inequality:
   $0 \leq 0/2 - 3.5$ is false. The region lies below the line and excludes origin (0, 0).

Simultaneous equations

Activity 1.16

Activity

Solve simultaneous equations

1. Solve these simultaneous equations both graphically and algebraically:

   a) $y = -x + 5, y = 5x - 7$

   b) $3y - x + 5 = 0, 2y = 2x - 1$

2. Discuss the possibility of solving these simultaneous equations:

   $y = -5x + 3.5, 2y + 10x + 6 = 0$

3. Discuss the possibility of solving these simultaneous equations:

   $3y = x + 9, -12y + 4x = -36$

4. Investigate whether it is possible to find a solution to these simultaneous equations:

   $y = -3x + 2, y = x - 4, 2y = -x + 5$

5. Summarise the results of questions 1– 4.

6. Describe, in terms of graphs, the implications of solving equations in three variables. You do not have to draw the actual graphs, but try to imagine them in space.

Solutions

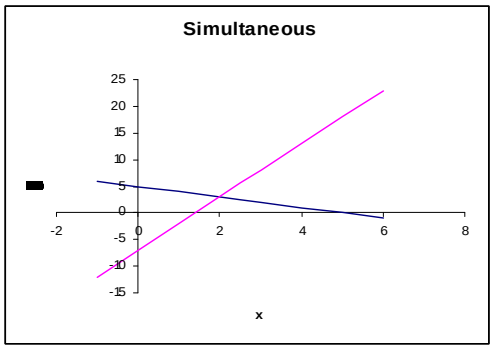
1. a) Algebraically: $-x + 5 = 5x - 7$

$$-6x = -12$$

$$x = 2$$

From $y = -x + 5$ we get $y = -2 + 5 = 3$

Graphically:



b) Algebraically: First, write in standard form.

$y = x/3 - 5/3$ and $y = 0.5x - 0.5$

Equate: $x/3 - 5/3 = 0.5x - 0.5$

$-x/6 = 7/6$

$x = -7$ and so $y = 0.5(-7) - 0.5 = -4$

Lecturers to review graph for correctness.

2. The equations are $y = -5x + 3.5$ and $y = -5x - 3$

   The slopes are the same and the lines therefore parallel. They do not intersect and there is no simultaneous solution.

3. The equations are $y = x/3 + 3$ and $y = x/3 + 3$

   The lines are identical. They do not intersect in a single point only but coincide everywhere. There are infinitely many solutions.

4. The lines intersect pair wise, but there is no point (x, y) satisfying all three equations.

5. Two straight lines either intersect in a unique point, or don't intersect (are parallel), or coincide. Two simultaneous linear equations in two variables, x and y, have a unique solution (x, y), no solution or infinitely many solutions. Three simultaneous linear equations in two variables have a unique solution only if two equations are identical and the third is different but not parallel.

6. Equations in three variables x, y and z represent planes in 3-D. Two planes intersect in a straight line (if they are not parallel or identical). Two equations in three variables therefore have infinitely many solutions, or no solution. However, three planes can intersect in a unique point (x, y, z).

Activity 1.17

Activity

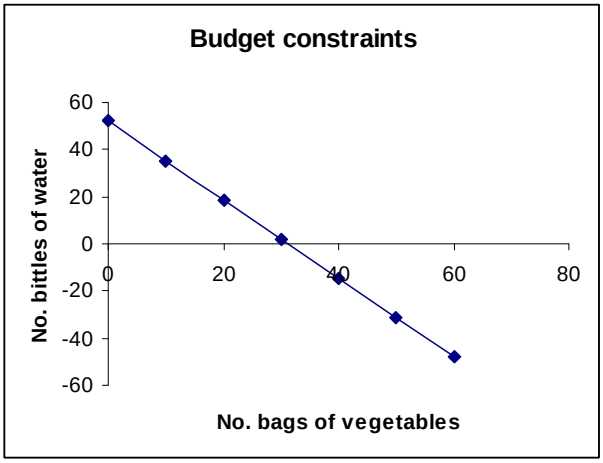
Budget constraint

What will you do?

1. Draw a graph for the budget constraint problem.
2. Solve the budget constraint problem for total budget $W = 10{,}000$ rupees.

Solutions

1. The model equation is $y = -1.67x + 52.08$, where x is the number of bags of vegetables and y the number of bottles of water.



It is now clear that we cannot buy more than 31 bags of vegetables. For $x = 40$, $y = -15$ approximately. This is impossible. We also cannot buy more than 52 bottles of water. The intercepts form the limits of what we can buy.

2. The budget constraint model is: $200x + 120y = 10{,}000$, which we re-write as $y = -1.67x + 83.33$. The analysis is similar to that in the Course Manual. If we buy no water, we can buy about 49 bags of vegetables. If we buy 30 bags of vegetables, we can buy $y = -1.67(30) + 83.33 = 33.23$, or 33 bottles of water.

Activity 1.18

Activity

Check the model

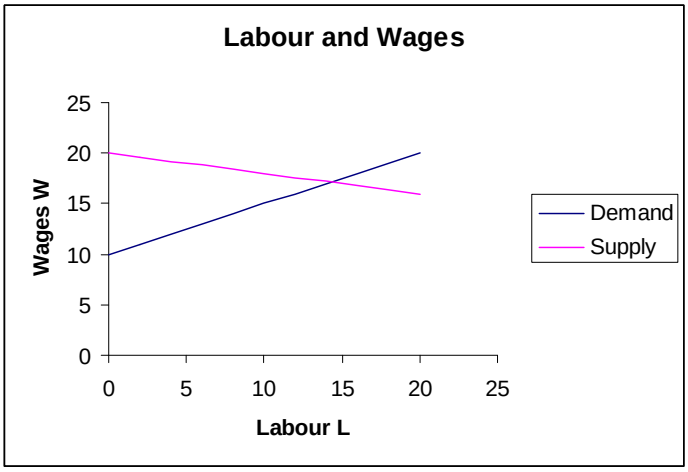
What will you do?

1. Interpret the results of the labour markets and wages model determined above.
2. Determine the equilibrium point graphically.

Solutions

1. Equilibrium or balance will be reached when the wages demanded by workers and wages offered by employers are equal. In this case, we see that 14 people can be employed. The corresponding wage per worker is 17 monetary units per period. One can also equate expressions for labour and solve for W first.

2.



Case study:
Break-even analysis

Firms that produce goods for sale have an income (revenue) from sales, but they also incur costs in the process. Generally, there are fixed costs and variable costs. Fixed costs are independent of the number of items produced, while variable costs are affected by the quantity of goods produced.

Assume that total costs equal fixed costs plus variable costs. We will also assume that variable costs are directly proportional to the number of items produced and nothing else, and that revenue comes only from selling items at a set price per unit.

Scenario: A small factory in India manufactures cell phone chargers and has determined that the price per unit will be 510.35 rupees (INR).

Fixed costs:	Rental	25,000
	Utilities	10,000
	Salaries	125,000
	Other	4,350
Variable costs:	Raw materials	0.6 INR × units produced
	Additional salaries	0.3 INR × units produced

Activity 1.19

Activity

Break-even report

What will you do?

Write a report on break-even for the factory that addresses these questions:

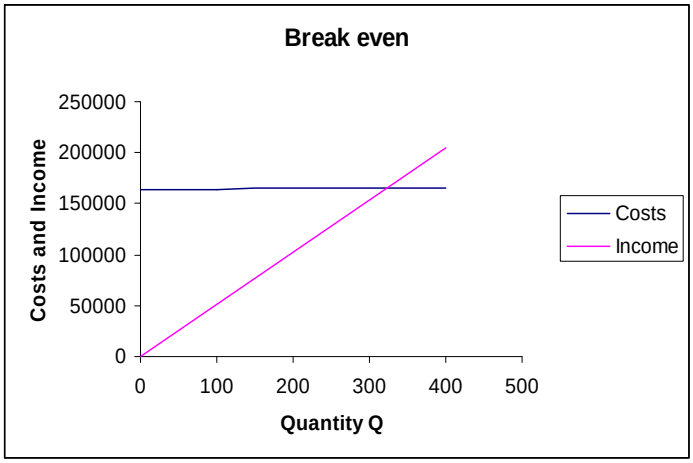
1. Denote total cost as C and number of items produced as Q. Derive the total cost equation $C(Q)$ for the factory.
2. Draw a graph of the total cost function.
3. Discuss the meaning of the values of the slope and intercept in this case.
4. Denote total income as I and derive the total income equation $I(Q)$ for the factory.
5. Graph the total income function.
6. Discuss the meaning of the values of the slope and intercept in this case.
7. Discuss and criticise the assumptions made to set up the model.
8. Define the meaning of the break-even point.
9. Calculate the break-even point algebraically and graphically.
10. Discuss your findings.

Solutions

1. Total cost $C(Q)$ = variable costs + fixed costs
   $= 0.9Q + 164{,}350$
2. See 9.
3. Intercept on C-axis is 164,350. This means that even if no items are produced, there are still costs. The slope of 0.9 indicates that for every item produced, the cost incurred goes up by 0.9 INR.
4. Total income $I(Q) = 510.35Q$
5. See 9.
6. Intercept on I-axis is 0. This means that even if no items are produced, there is no income. The slope of 510.35 indicates that for every item produced, the income goes up by 510.35 INR.
7. The cost and income functions are very simple (linear) and do not take risk factors into account. We are also assuming number of units produced equals number of units sold.
8. Break-even is where costs and income are equal: $C(Q) = I(Q)$. At that stage, the company makes no profit but merely covers costs. It is important to know the value of Q at break-even so that we know how many units to produce (sell) to cover costs.
9. Algebraically: Set $C(Q) = I(Q)$, i.e. $0.9Q + 164{,}350 = 510.35Q$

This yields $Q = 322.6$ units. We should, therefore, produce at least 323 units to break even.

Graphically:



10. At 323 units, the income will be 164,843.05 INR. When we increase Q above 323, we see the Income line rising above the Costs line and a profit is made.

Unit 3

Non-linear functions and equations

Introduction

Quadratics, the exponential and logarithmic functions are important in statistics, business and finance, Probability theory and Utility theory.

Upon completion of this unit students will be able to:

Outcomes

- **solve** quadratic functions;
- **sketch** quadratic functions;
- **apply** quadratic functions in management problems; and
- **recognise** and use other non-linear functions (exponentials and logarithms).

Quadratics and their graphs

Activity 1.22

Activity

Draw some more

What will you do?

1. Consider the equation $y = 2x^2 + 3x - 2$
   a) Draw up a table of x and y values.
   b) Find the y-intercept.
   c) Calculate the value of Δ and interpret your answer.
   d) Draw the graph of the quadratic.
   e) Give the values of x and y where the quadratic reaches its maximum.

Solutions

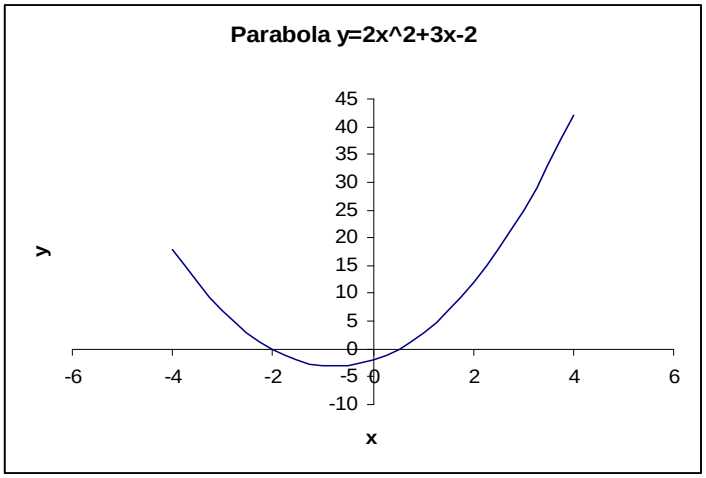
a)

x-values	y-values
−3	7
−2	0
−1	−3
0	−2
1	3
2	12
3	25

b) y-intercept = −2

c) $\Delta = b^2 - 4ac = 3^2 - 4(2)(-2) = 25 > 0$. The graph will intersect the x-axis in two places. These are the roots of the equation or the zeroes of the quadratic. The roots are: $x = \dfrac{-b-\sqrt{\Delta}}{2a}$ and $x = \dfrac{-b+\sqrt{\Delta}}{2a}$, i.e.
$x = -2$ and $x = 0.5$

d) The parabola extends indefinitely.



e) The quadratic does not reach a maximum. There is a minimum at $x = -0.75$ where $y = -3.125$. This is the turning point of the parabola.

Exponentials, logarithms and their graphs

Activity 1.23

Activity

Solve logarithms

1. Use a pocket calculator to determine the values of:
   a) e^3
   b) e^4
   c) e^7 and compare this to $e^3 e^4$
2. Complete the table and discuss.

x-values	x^3-values	3^x values
0		
1		
2		
3		
4		

3. Solve for n if:
   a) $(1.1)^n = 40$
   b) $20\,(1 + 0.05)^n = 10$

Solutions

1. a) $e^3 = 20.0855$ (rounded off)
   b) $e^4 = 54.5982$ (rounded off)
   c) $e^7 = 1096.6332$ (rounded off) $= e^3 \times e^4$
2.

x-values	x^3-values	3^x values
0	0	1
1	1	3
2	8	9
3	27	27
4	64	81

The exponential function 3^x grows faster than the cubic x^3.

3. a) Apply the ln function: $\ln(1.1)^n = \ln 40$

$n \ln 1.1 = \ln 40$

$n = \ln 40 / \ln 1.1$

$n = 38.7039$

b) Apply the ln function: $\ln [20 (1 + 0.05)^n] = \ln 10$

$\ln 20 + \ln(1 + 0.05)^n = \ln 10$

$n \ln(1.05) = \ln 10 - \ln 20$

$n = -14.21$ (rounded)

Apply non-linear functions

Case study:
Supply and demand

The supply and demand for a product manufactured in your factory is described by the following functions:

$P_S = 2Q_S^2 + Q_S - 4$ and $P_D = Q_D^2 - 2Q_D + 3$

Quantities are in hundreds and prices in thousands of units of your country's currency.

Separately for each function, determine the minimum price and quantities acceptable to buyers and to yourself.

Then determine the market equilibrium and discuss your findings.

Solution

Demand side: Minimum price P_D is where the parabola has a turning point at $Q_D = 1$. ($x = \frac{-b}{2a}$). Min $P_D = 2$. Therefore, a minimum demand of 1,000 at price 2 per unit.

Supply must be greater than 1,000 for a positive price. (Min P_S is at $Q_S = -1/4$ which, of course, is meaningless.)

Equilibrium: $2Q_S^2 + Q_S - 4 = Q_D^2 - 2Q_D + 3$ and $Q_S = Q_D = Q$

$Q^2 + 3Q - 7 = 0$

$$Q = \frac{-3 - \sqrt{37}}{2} \text{ and } Q = \frac{-3 + \sqrt{37}}{2}$$

Equilibrium quantity = 1.541 (rounded down) 1,541 units of the product.
Equilibrium price = 2.29 monetary units per item.

If more units are produced and sold, the supply price goes up. This can occur when there is a greater demand for the product (perhaps created through marketing). At $Q = 3$ the supply price is $P_S = 17$ and the demand price is $P_D = 6$. However, good marketing can change the shape of the demand price function, say to $Q_D^2 - 2Q_D + 16$.

Unit 4

Differentiation and integration

Introduction

Calculus is the part of mathematics that studies changes in quantities. Rate of change is described by the process of differentiation, and integration is the "reverse" process of differentiation. Integration can also be seen as a process of summing and averaging.

Calculus is now done at secondary level in many schools and is no longer considered as "advanced" mathematics.

Upon completion of this unit students will be able to:

Outcomes

- **differentiate** simple functions;
- **calculate** rates of change of quantities;
- **perform** basic integration;
- **determine** maxima and minima of quantities; and
- **apply** differentiation and integration in management problems.

Differentiation

Differentiation is the tool for optimisation of quantities. We present the basics only. Lecturers may add to the notes if they wish.

Activity 1.26

Activity

Solve derivatives

What will you do?

1. Find the derivatives $\frac{dy}{dx}$ if:

   a) $y = 5x + 6$

   b) $y = -0.3x + 2$

   c) $y = 2x^2 + 4x + 3$

   d) $y = -0.8\,x^2 - x + 12$

   e) $y = 6x^3 - 0.5x^2 + 3x + 4$

2. Explain the meaning of the equation $\frac{dy}{dx} = -3$

   What relationship can be expected between y and x?

3. Explain the meaning of the equation $\frac{dy}{dx} = 2x + 4$

   What relationship can be expected between y and x? If the relationship is expressed graphically, what is the rate of change of y with respect to x at $x = 4$?

4. Quantities P and Q are related by the equation $2P = 6Q^2 - Q + 4$
   If $Q = 1$, what is the value of P? What is the rate of change of P with respect to Q at this point?

5. What is the slope of the line describing the relation
   $5x - 3y - 12 = 0$?

Solutions

1. a) $\frac{dy}{dx} = 5$ b) $\frac{dy}{dx} = -0.3$ c) $\frac{dy}{dx} = 4x + 4$

   d) $\frac{dy}{dx} = -1.6x - 1$ e) $\frac{dy}{dx} = 18x^2 - x + 3$

2. The slope of the graph of $y(x)$ is constant and negative. As x increases, y will decrease. For every unit increase in x, y decreases by 3 units. The relationship is linear.

3. The slope of the graph of $y(x)$ changes: it increases from negative for $x < -2$ to positive for $x > -2$. The slope is zero at $x = -2$, which indicates a turning point at $x = -2$. The function $y(x)$ must be a quadratic. In fact, $y = x^2 + 4x + k$ for some constant k.

   The rate of change at $x = 4$ is $\frac{dy}{dx} = 2x + 4$ (at $x = 4$) $= 12$

4. $P(Q) = 3Q^2 - 0.5Q + 2$, so $P(1) = 3(1)^2 - 0.5(1) + 2 = 4.5$

   $\frac{dP}{dQ} = 6Q - 0.5 = 5.5$ at $Q = 1$

5. Write: $5x - 3y - 12 = 0$ in the form $y = (5/3)x - 4$. Then $\frac{dy}{dx} = 5/3$

Optimisation: Maxima and minima

One of the most important applications of derivatives in decision science is to find optimal values of quantities.

Activity 1.28

Activity

Use more calculus methods

What will you do?

Use the methods of calculus to answer these questions:

1. Consider function $y = 4x - 2$ with constraint $x \geq -1$.
   a) Is the function increasing or decreasing? Justify your answer.
   b) Find the minima and maxima (if they exist).
   c) Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
2. Consider function $y = -x^2 + 2x - 3$
   a) Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
   b) Is the function increasing or decreasing? Justify your answer.
   c) Find the minima and maxima (if they exist).
   d) If you add the constraint $-2 \leq x \leq 1$, how does this change your answer for 2c)?
3. A certain function T has a turning point at $s = 6$.
   The value of $\frac{dT}{ds}$ is $\frac{dT}{ds} = s^2 - 36$
   Does the function have a minimum or maximum at $s = 6$?
4. If $\frac{dP}{dQ} = -2x + 5$, can you guess what type of relationship may exist between P and Q?

Solutions

1. a) Linear function with positive slope $m = 4$. Increasing from $x = -1$.
   b) There is no maximum. There is a minimum at $x = -1$: Minimum value is $y = -6$.
   c) $\frac{dy}{dx} = 4$ and $\frac{d^2y}{dx^2} = 0$
2. a) $\frac{dy}{dx} = -2x + 2$ and $\frac{d^2y}{dx^2} = -2$
   b) The function has the shape ∩ and increases for $x < 1$ where $\frac{dy}{dx} > 0$ and decreases for $x > 1$ where $\frac{dy}{dx} < 0$

c) $\frac{d^2y}{dx^2} < 0$ means there is a maximum where $\frac{dy}{dx} = 0$. Setting $-2x + 2 = 0$ yields $x = 1$. The maximum is at turning point (1, −2). Since the graph can be extended indefinitely, there is no minimum.

d) The constraint means the parabola is cut off at $x = -2$ and at $x = 2$. There are now two minima at $x = -2$ and at $x = 2$. The minimum value points are (–2, −11) and (2, −3).

3. Calculate $\frac{d^2T}{ds^2} = \frac{d}{ds}[s^2 - 36] = 2s$. At $s = 6$ we have $\frac{d^2T}{ds^2} = 12 > 0$ and so T has a minimum at $s = 6$.

4. Quadratic.

Integration

Integration can, at a simplified level, be seen as the reverse operation to differentiation. It is used extensively in probability density functions for continuous random variables such as the normal distribution.

Indefinite integrals

$$\int \frac{dy}{dx}dx = y(x) + K$$

Activity 1.30

Activity

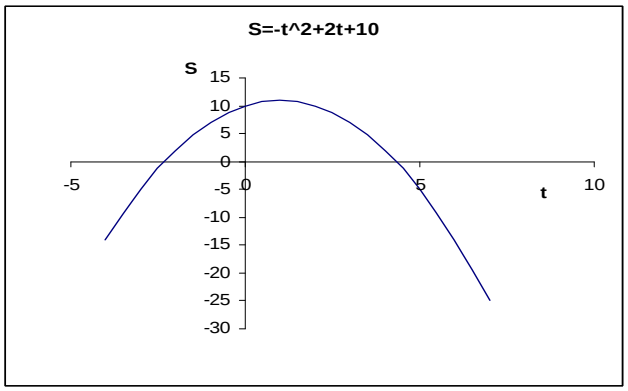
Calculate the integrals

What will you do?

1. $\int (2x - 2)dx$
2. $\int (-x + 1)dx$
3. $\int (4 - 4x)dx$
4. $\int (x^2 + 3x - 1)dx$
5. Find $P(Q)$ if $\frac{dP}{dQ} = 5$ and it is given that the graph has P-intercept equal to 4. Draw the graph of P.
6. Find $S(t)$ if $\frac{dS}{dt} = -2t + 2$ and it is given that the graph has S-intercept equal to 10. Draw the graph and determine the maximum value of S.

Solutions

1. $x^2 - 2x + K$
2. $-\frac{1}{2} x^2 + x + K$
3. $4x - 2x^2 + K$
4. $\frac{1}{3} x^3 + (3/2)x^2 - x + K$
5. $P(Q) = 5Q + K$. We are given $P(0) = 4$, but $P(0) = K$ and so $K = 4$. Therefore $P(Q) = 5Q + 4$
6. $S(t) = -t^2+2t + K$ and $S(0) = 10$. Therefore $S(t) = -t^2+2t + 10$. This is a parabola with shape ∩. The maximum is where $\frac{dS}{dt} = -2t + 2 = 0$. This is where $t = 1$. Maximum value is $S(1) = 11$.



Definite integrals

Definite integrals have the form $\int_a^b y dx$. The numbers a and b are the bounds for the integral.

$$\int_a^b \frac{dy}{dx} dx = y(x) \Big|_{x=a}^{x=b} = y(b) - y(a)$$

Activity 1.31

Activity

Calculate and interpret

What will you do?

Calculate $\int_{-1}^{1} x dx$ and interpret your answer.

Solution

$$\int_{-1}^{1} x\,dx = \tfrac{1}{2}x^2\Big|_{x=-1}^{x=1} = \tfrac{1}{2}(1)^2 - \tfrac{1}{2}(-1)^2 = 0$$

The area of the triangular region between the graph of $y = x$ and x-axis between $x = -1$ and $x = 0$ is 0.5 and equals the area of the triangle below the x-axis between $x = 0$ and $x = 1$. The "net total area" between the graph of $y = x$ and the x-axis between $x = -1$ and $x = 1$ is thus zero!

Assessment

Assessment — Module 1

1. If the amount $10,500.00 includes tax of 12%, what was the original amount excluding tax?
2. What is the rate of return per annum on an investment with value that increases from 12,000 to 13,634 over the year?
3. You are given the sequence 2, 3, 4.5, 6.75…
   a) Find the formula for the general term of the sequence.
   b) Determine the sum of the first 20 terms of the series.
4. An investment of 10,000 cedi has a cash inflow of 4,500 cedi at the end of each year for the next three years. The discount rate is 4.5%.
   a) Calculate the present value of all cash flows.
   b) Determine the *NPV* and *PI* for the investment. Discuss your findings.
   c) Do a sensitivity analysis for a scenario where you expect the interest rate to go up.
5. Draw the graph and give the equation for the linear relationship between quantities P and Q if you are given the two data points (P, Q): (22, 4) and (46, 12). Take Q as the dependent variable.
6. Solve these simultaneous equations both graphically and algebraically:

   $2y - 2x + 6 = 0$ and $2y = x - 5$

   Now simultaneously solve the inequalities $2y - 2x + 6 \geq 0$ and $2y \geq x - 5$
7. A cell phone stall allows customers to make calls of three minutes' duration. Let Q denote the average number of phone calls made per week. Costs are denoted in units of dollars.

   The total weekly cost function for supplying the service is $C = 3Q + 50$

   The total weekly income function for this service is $I = 100Q - 0.5Q^2$

   Analyse the situation for break-even and profit, following the method in Unit 4.
8. Find the maximum value of $f(x)$ if $f(x) = -3x^2 + 5x - 1$
9. Determine the value of the definite integral and interpret it:

   $\int_{-2}^{2} (-x^2 + 4)dx$
10. Let the total cost of production be y and let x denote the number of items manufactured. The marginal cost is MC $= \frac{dy}{dx}$

A small factory in Malaysia has produced 200 items. It would like to expand its operations and produce 300 items, but only has 20,000 rupees available to fund expansion. It is known that marginal cost MC = $\frac{dy}{dx} = x - 4$ at this stage. What would be the total cost for the next 100 items to be manufactured? Can the factory afford it?

Solutions

1. If original amount is X, then $X(1+0.12) = 10{,}500.00$ and $X = 9{,}375.00$.

2. Let the rate be R. Then $12{,}000(1 + R) = 13{,}634$

$$(1 + R) = 1.1362$$

$$R = 0.1362 = 13.62\%$$

3 a) $T_n = 2(1.5)^{n-1}$, n=1, 2, 3...

b) This is a geometric sequence with ratio $r = 1.5$, so:

$$S_n = \frac{T_1(1-r^n)}{1-r} =. \frac{T_1(r^n-1)}{r-1}$$

$$S_{20} = \frac{2(1.5^{20}-1)}{1.5-1} = 13{,}297.03$$

4. a) $$PV = \frac{4500}{(1+0.045)} + \frac{4500}{(1+0.045)^2} + \frac{4500}{(1+0.045)^3}$$
= 12,370.34 cedi

b) $NPV = PV - 10{,}000 = 2{,}370.34$ cedi
$PI = 1.237$
The investment is profitable ($NPV > 0$, $PI > 1$) as long as there is no risk.

c) Assume the discount rate goes up to 8%. Then:

$$NPV = \frac{4500}{(1+0.08)} + \frac{4500}{(1+0.08)^2} + \frac{4500}{(1+0.08)^3} - 10{,}000$$

= 11,596.94 – 10,000 = 1,596.94 (> 0)

The investment is still profitable (although less so). Even at 12% $NPV > 0$. The investment seems quite stable under changes in interest rate.

5. $m = \frac{8}{24} = \frac{1}{3}$ so $Q = \frac{1}{3}P + c$. Substitute (22, 4): $4 = \frac{22}{3} + c$

Linear relation: $Q = \frac{1}{3}P - \frac{10}{3}$

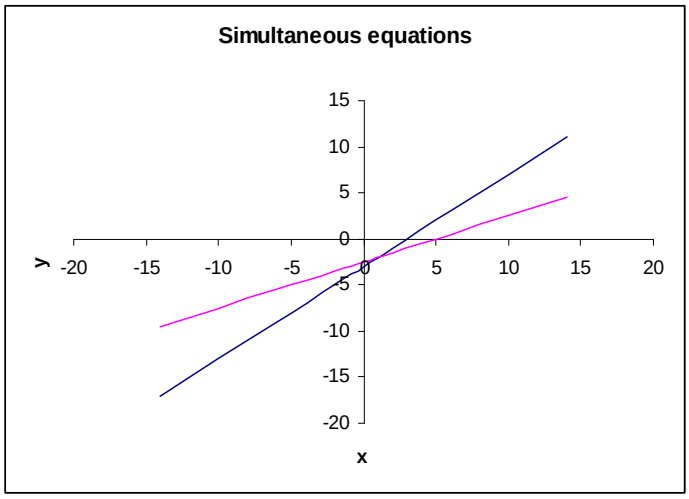
The graph has slope $\frac{1}{3}$ and Q-intercept $-\frac{10}{3}$

Lecturers should review graph for correctness.

6. Rewrite equations: $y = x - 3$ $y = 0.5x - 2.5$

   $x - 3 = 0.5x - 2.5$ implies $0.5x = 0.5$ and $x = 1$

   Point of intersection: $(1, -2)$



   $(0, 0)$ lies in both regions $2y - 2x + 6 \geq 0$ and $2y \geq x - 5$. The region described by the inequalities simultaneously is the intersection of the areas above the two lines.

7. Break-even: $C = 3Q + 50$ must equal $I = 100Q - 0.5Q^2$

   $3Q + 50 = 100Q - 0.5Q^2$

   This can be written as: $Q^2 - 194Q + 100 = 0$

$$= \frac{-b - \sqrt{\Delta}}{2a} \text{ or } Q = \frac{-b + \sqrt{\Delta}}{2a}$$

   $Q = -200$ or $Q = 193.48$

   At 193 calls per week, they will have break-even. To make a profit, they need customers to make more than 194 calls per week.

8. Maximum is where $\frac{df}{dx} = -6x + 5 = 0$; i.e. $x = 5/6$

   Max $f(x) = -3(5/6)^2 + 5(5/6) - 1 = 1.083$

9. $\int_{-2}^{2} (-x^2 + 4)dx$

$$= [-\tfrac{1}{3}x^3 + 4x]\Big|_{x=-2}^{x=2} = [-\tfrac{1}{3}(2)^3 + 4(2)] - [-\tfrac{1}{3}(-2)^3 + 4(-2)]$$

$$= 16/3 - [-16/3] = 32/3$$

   It is the total area between the graph of $(-x^2 + 4)$ and the x-axis between $x = -2$ and $x = 2$.

10. MC = $\frac{dy}{dx} = x - 4$

Additional cost = $\int_{200}^{300}(x-4)dx$

$= [½x^2 - 4x]\Big|_{x=200}^{x=300} = [½(300)^2 - 4(300)] - [½(200)^2 - 4(200)]$

= 24,600 rupees

The cost of expansion is slightly more than the factory can afford.

Module 2

Statistical tools and probability theory

Outcomes

Upon completion of this module the student will be able to:

- **display** data in the form of graphs and charts;
- **identify** appropriate statistical tools in analysing data (mean, median, and so on);
- **interpret** results correctly to inform decisions;
- **understand** probabilities of events;
- **handle** probability distributions and understand their role in modelling; and
- **apply** expected values, variance and standard deviation to risk and volatility.

Unit 5

Collecting and presenting data

The basis of decision-making lies in data and information. Data have to be analysed and interpreted before we can present it as information and use it to make management decisions.

Upon completion of this unit the student will be able to:

Outcomes

- **classify** different types of data;
- **represent** data in tables;
- **display** data in charts and graphs; and
- **draw** and **display** frequency tables.

Data collection and tables for data

Activity 2.1

Activity

Interpret graphs

What will you do?

1. Calculate the size of slice each colour makes in the pie chart in Figure 6 and show that the slices add up to 360˚.
2. Represent the car colour data as a column chart.

Solutions

1. Black: 45% of 360˚ = 162˚

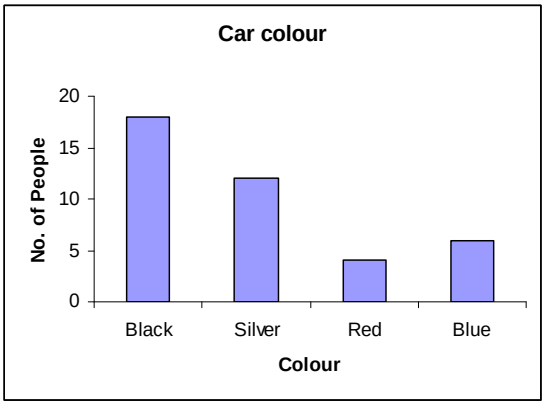
   Silver: 30% of 360˚ = 108˚

   Red: 10% of 360˚ = 36˚

   Blue: 15% of 360˚ = 54˚

   The percentages add up to 100% and the angles add up to 360˚.

2.



Frequency distributions: tables and histograms

Activity 2.2

Activity

Make graphs and tables

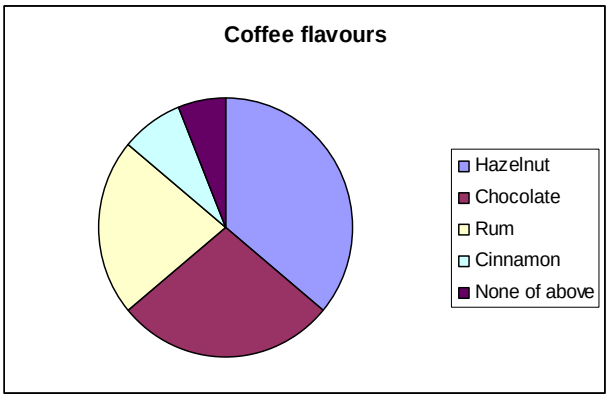
What will you do?

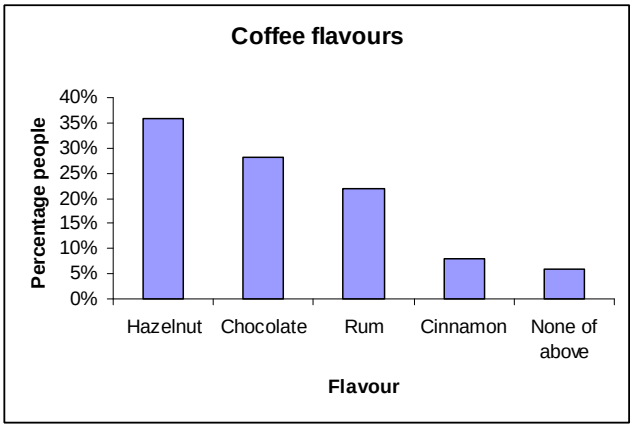
1. You want to conduct a survey to find out which new flavour of coffee will sell well in your country. The available flavours are chocolate, hazelnut, cinnamon and rum. Describe how you would go about gathering data.
2. Your supplier in another country has this data for her own country:
   - 36% of people preferred hazelnut,
   - 28% preferred chocolate,
   - 22% preferred rum
   - 8% preferred cinnamon, and
   - the remainder didn't like any of the flavours.

   Present the supplier's findings graphically in two ways. What conclusion can you draw?
3. Draw a frequency table, histogram and ogive for the following data. Of 2,000 adults surveyed:
   - 8% had a mass between 120 kg and 140 kg,
   - 20% had a mass between 100 kg and 120 kg,
   - 36% had a mass between 80 kg and 100 kg,
   - 28% had a mass between 60 kg and 80 kg, and
   - the remainder had a mass between 40 kg and 60 kg.

Solutions

1. Surveys can be done by post, email, telephone, interviewing people in coffee shops, and so on. Data can be presented in tables, column charts, pie charts, and so on.
2. Column and pie charts can be used.



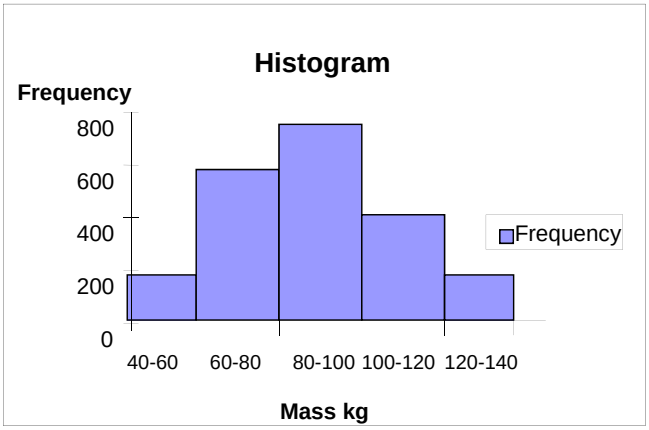


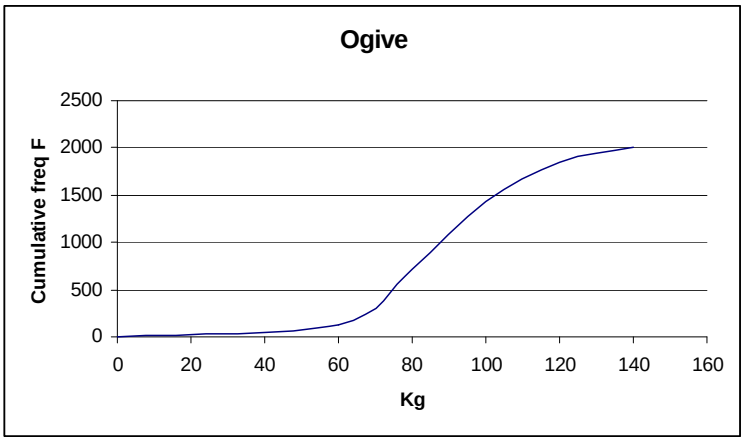
The disadvantage of the pie chart is that the percentages are not stated. It may be difficult to distinguish between the popularity of chocolate versus rum flavours. Percentage data also does not indicate how large the sample was.

3.

Mass in kilograms (x)	Number of people (frequency f(x))	Cumulative frequency F(x)
40–60	160	160
60–80	560	720
80–100	720	1440
100–120	400	1840
120–140	160	2000

Histogram


Histogram
Frequency
800
600
400
200
0
40-60
60-80
80-100
100-120
120-140
Mass kg
Frequency


Ogive
Cumulative freq F
2500
2000
1500
1000
500
0
0
20
40
60
80
100
120
140
160
Kg

Unit 6

Measures for analysing data

Outcomes

Upon completion of this unit students will be able to:

- **calculate** the mean value of discrete data;
- **calculate** the median and mode of discrete data;
- **determine** the mean and median for grouped data with frequency distributions;
- **apply** different measures of spread or dispersion of data;
- **compare** different data sets using co-variance or correlation; and
- **critique** some uses of statistics.

Mean, median and mode

Activity 2.5

Activity

Practise some more

1. Find the mean, median and mode of this set of data:
   5, 2, 6, 4, 10, 7, 9, 11, 4, 7, 8, 3

   Represent the data in a scatter graph.
2. Study the frequency table for continuous data:

Class of values (x)	Frequency $f(x)$
2.0–4	4
4.0–6	7
6.0–8	15
8.0–10	10
10.0–12	2

Figure 2

a) Expand the table to include cumulative and relative frequencies.

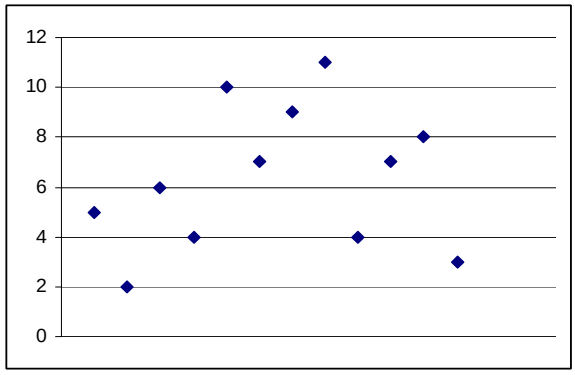
b) Find the mean and median for the distribution.

c) Draw a histogram and ogive.

d) Use the ogive to confirm the value of the median.

Solutions

1. Mean = 6.33 (rounded), median = 6.5, modes = 4 and 7



2. a)

Class of values (x)	Frequency $f(x)$	Cumulative frequency F	Relative frequency
2.0–4	4	4	0.1053
4.0–6	7	11	0.1842
6.0–8	15	26	0.3947
8.0–10	10	36	0.2632
10.0–12	2	38	0.0526

a) Midpoints x_i of classes: 3, 5, 7, 9, 11

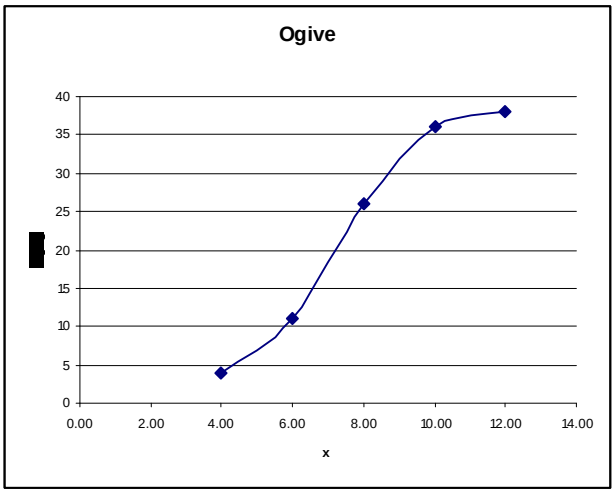
$$\text{Mean} = \frac{1}{N}\sum_{i=1}^{N} x_i f(x_i) = \frac{1}{38}[3(4) + 5(7) + 7(15) + 9(10) + 11(2)]$$

= 6.95 (rounded)

The middle values are observations 19 and 20, which fall in class 6–8. Since 19 – 11 = 8 and 20 – 11 = 9, the middle value takes position between 8 and 9 of the 15 values in class 6–8. Now 6 + (8/15)2 = 7.07 and 6 + (9/15)2 = 7.2 so the median is ½(7.07 + 7.2) = 7.135.

c) Straightforward

d)



The midpoint of values on the vertical axis is 17. The corresponding x-value is about 7.

Skewness and spread of data

Case study:
Investment decisions

You are analysing the daily price movements of two shares, with a view to making a recommendation about buying one, or both, of the shares.

These numbers are the share prices (in dollars) of shares S1 and S2 collected daily at the close of the trading day for 21 days in February 2009.

Day	1	2	3	4	5	6
S_1	12,00	12,15	12,85	14,00	13,80	13,80
S_2	37,00	36,20	35,00	34,80	38,00	44,50

Day	7	8	9	10	11
S_1	11,00	10,40	10,00	13,00	15,50
S_2	46,00	47,80	45,00	42,00	39,80

Day	12	13	14	15	16
S_1	16,50	16,00	15,00	14,50	14,20
S_2	35,00	48,00	52,40	54,00	54,20

Day	17	18	19	20	21
S_1	13,00	12,20	13,00	14,00	14,20
S_2	56,00	58,20	52,40	51,00	50,00

The daily rate of return for S_1 at the end of day two is calculated as:

$$\frac{12{,}15 - 12{,}00}{12{,}00} = 0{,}0125 = 1{,}25\%$$

Similarly, the daily rate of return for S_2 for day 10 is:

$$\frac{42{,}00 - 45{,}00}{45{,}00} = -0{,}067 = -6{,}7\%$$

1. Draw a table to show the daily prices and the 20 daily rates of return for each share.
2. Use graphs to present your data in an attractive way.
3. Find the mean daily rates of return over the month for each share.
4. Calculate the one-day standard deviation of rates of return for each share over the month. Use this to estimate the monthly volatility of each share.
5. Calculate the correlation between the two share price processes, as well as the correlation between the returns for the month.
6. Do some research on the topic of "diversification in investments" and write a short report on this. Show how diversification is linked with correlation of share prices.
7. Write a short paragraph comparing the two shares in terms of expected returns and volatility. What recommendations will you make for investment in these shares?

Solutions

1.

ReturnR1	ReturnR2
0.0125	-0.02162
0.057613	-0.03315
0.089494	-0.00571
-0.01429	0.091954
0	0.171053
-0.2029	0.033708
-0.05455	0.03913
-0.03846	-0.05858
0.3	-0.06667
0.192308	-0.05238
0.064516	-0.1206
-0.0303	0.371429

ReturnR1	ReturnR2
-0.0625	0.091667
-0.03333	0.030534
-0.02069	0.003704
-0.08451	0.03321
-0.06154	0.039286
0.065574	-0.09966
0.076923	-0.02672
0.014286	-0.01961

2. A variety of graphs can be drawn. Students can graph daily share prices or daily rates of return.

3.

Mean return	0.013508	0.020049

4.

Std dev return	0.10576	0.107628
cor(S1,S2)	-0.04361	
cor(R1,R2)	-0.39843	

5. The shares are negatively correlated. The combination of S1 and S2 gives some diversification in a portfolio and protection. Information on diversification is available on the Internet.

6. Negative correlation means as one variable tends to rise above its mean, the other tends to fall below its mean. In portfolio analysis, this provides protection by not having all prices rise and fall together (positive correlation). This is what is meant by diversification. One should try to include a number of shares in a portfolio, some of which should be negatively correlated.

7. Share S2 has a higher daily rate of return and a slightly higher risk (volatility or standard deviation of return) than Share S1.

   Recommendation: Based on this risk-return analysis only, it is worthwhile investing in Share S2. The increase in risk is very small (0.2%) for an increase in return of about 0.7% as compared to Share S1.

Unit 7

Basic probability theory

Outcomes

Upon completion of this unit students will be able to:

- **describe** outcomes and events for experiments;
- **calculate** probabilities for outcomes and events;
- **understand** conditional probability and Bayes' formula; and
- **identify** mutually exclusive and independent events.

Dependent, independent and mutually exclusive events

Activity 2.9

Activity

Calculate more probabilities

1. Event E has probability 0.65 and event F has probability 0.3.
   a) If the sample space Ω has 200 outcomes, how many observations are there in each event?
   b) 1.2. What is $P(E^c)$? Calculate $P(E \cup E^c)$ and discuss your answer. Calculate $P(E \cap E^c)$ and discuss your answer.
   c) If E and F are independent, determine $P(EF)$.
2. You have 120 staff members: 86 female and 34 male. There are 5% of staff members who are disabled. What is the probability that a staff member is female and disabled?
3. What is the total number of outcomes in having to choose any three letters from the English alphabet? What is the total number of outcomes in choosing three letters from the English alphabet if you may only choose each letter once?
4. A lottery requires you to pick five different numbers, each from the set of numbers from 1 to 20. What is the probability of picking the correct number? How does the probability change when you can choose from numbers 1 to 50?
5. The share price today is USD 20. At the end of the week it will go either up by 15% with probability 0.3 or down by 10% with probability 0.7 respectively. At the end of the following week the then price will go either up or down by 10% with probabilities 0.5 and 0.5, respectively. Draw a probability tree. What is the probability that the share price will be USD 19.80 at the end of the second week?

Solutions

1. a) E: 130 observations F: 60 observations

   b) $P(E^c) = 0.35$ $P(E \cup E^c) = 1$ $P(E \cap E^c) = 0$

   $E \cup E^c = \Omega$ and $P(\Omega) = 1$ $E \cap E^c =$ the empty set

   c) $P(EF) = P(E)\,P(F) = 0.65 \times 0.3 = 0.195$

2. Let F be the event staff member is female. Let M be the event staff member is male. Let D be the event staff member who is disabled. $P(F) = 86/120 = 0.7167$ $P(M) = 0.2833$ $P(D) = 0.05$ (given)

   $P(F \text{ and } D) = P(FD) = P(F)\,P(D)$ (female and disabled are independent)

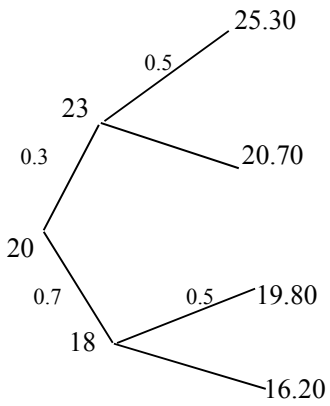
   $= 0.7176\ (0.05) = 0.03588$

3. Total number of outcomes = $26 \times 26 \times 26 = 17{,}576$ (each slot can contain any of 26 letters). If each letter can be chosen only once, the number of outcomes is $26 \times 25 \times 24 = 15{,}600$ (once a letter has been picked it is no longer available).

4. The number of possible choices is: $20 \times 19 \times 18 \times 17 \times 16 = 1{,}860{,}480$

   The probability of choosing the correct number is 1/1,860,480 or 0.00000054

   If you choose from numbers 1 to 50, the probability is 0.000000004!

5. 

   The probability that the share price will be 19.80 at the end of two weeks is:
   P(price goes down first *and* then up) = $0.7 \times 0.5 = 0.35$

Conditional probability and Bayes' formula

Knowledge of the occurrence of one event can affect the probability of the other event.

Activity 2.10

Activity

Consolidate probability

What will you do?

1. Suppose events E and F are independent, with $P(E) = 0.3$ and $P(F) = 0.55$. Calculate:
   a) $P(E \cup F)$
   b) $P(E \mid F)$
2. You have two managers in your division. What is the conditional probability that both are female (f), given that at least one of them is female?

Hint: The sample space is $\Omega = \{(f, f), (f, m), (m, f), (m, m)\}$. Let E be the event that both are female and F the event that at least one of them is female.

3. Your staff are given a competency test in the form of a multiple-choice test. Each question has four answers to choose from. You want to evaluate their performance on a specific question. A person either knows the answer to this question or guesses. Assume that the probability of a person knowing the answer is 0.5 and the probability of them guessing is 0.5. Assume also that the probability that a person who guesses gets the answer correct is ¼.

   What is the conditional probability that a person really knew the answer to the question, given that she or he answered it correctly?

Hint: Use Bayes' formula, with E being the event that they actually knew the answer and F the event that they gave the correct answer.

Solutions

1. a) $P(E \cup F) = P(E) + P(F) - P(EF)$
   $= P(E) + P(F) - P(E)\,P(F)$ (independence)
   $= 0.3 + 0.55 - (0.3)(0.55)$
   $= 0.685$

   b) $P(E \mid F) = \dfrac{P(EF)}{P(F)} = \dfrac{0.165}{0.55} = 0.3 = P(E)$

2. $E = \{(f, f)\}$ and F = $\{(f, f), (f, m), (m, f)\}$

   $P(E \mid F) = \dfrac{P(EF)}{P(F)}$ where $P(E) = \dfrac{1}{4}$ and $P(F) = \dfrac{3}{4}$

   Now $P(EF) = \dfrac{1}{4}$ since the intersection $E \cap F = \{(f, f)\}$.Therefore

   $P(E \mid F) = \dfrac{0.25}{0.75} = \dfrac{1}{3}$

3. F is the event that a person gives the correct answer to the question. E is the event that the person actually knows the answer to the question and E^c is the mutually exclusive event that they guess. We know $P(E) = 0.5$, $P(E^c) = 0.5$, $P(F \mid E^c) = 0.25$ and so $P(F \mid E) = 0.75$

   We must find $P(E \mid F)$.

   Bayes' formula states that:

   $$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^c)P(E^c)}$$

   $$= \frac{0.75(0.5)}{0.75(0.5) + 0.25(0.5)} = 0.75$$

Unit 8

Probability distributions and applications

Outcomes

Upon completion of this unit the student will be able to:

- **identify** discrete random variables and their probability mass functions;
- **apply** the binomial distribution and Poisson distribution;
- **identify** continuous random variables;
- **understand** the normal and uniform distributions and their density functions; and
- **apply** the normal distribution in market research by **calculating** confidence intervals and **applying** hypothesis tests.

Probability distributions and applications

Activity 2.14

Activity

Revisit probability

What will you do?

1. A machine produces items with a 10 per cent probability of being defective. What is the probability that in a sample of three items, at most one will be defective?
2. The number of accidents occurring on a road each week has a Poisson distribution with mean 6. Determine the probability that there will be 10 accidents the following week.
3. A continuous random variable X has probability density function $f(x) = 0.5$ for $0 < x < 2$, and $f(x) = 0$ for all other x.

   Calculate $\overline{X}$ and $\sigma^2(X)$.
4. Returns on assets are assumed to be normally distributed, with mean value 20 per cent and standard deviation $\sigma = 40$ per cent.

   What is the probability that the return will be:

   a) greater than 20 per cent?

   b) between −20 per cent and 60 per cent?

Solutions

1. X is the number of defective items per sample of three. Use the binomial distribution with $p = 0.10$ and $N = 3$. The probability that at

most one item is defective is: $P(X = 0 \text{ or } X = 1) = P(X = 0) + P(X = 1)$
$= p(0) + p(1)$

Now $p(0) = \binom{3}{0}(0.1)^0(1-0.1)^{3-0} = 1\times1\times(0.9)^3 = 0.729$

$p(1) = \binom{3}{1}(0.1)^1(1-0.1)^{3-1} = 3\times(0.1)\times(0.9)^2 = 0.243$

Therefore $P(X = 0 \text{ or } X = 1) = 0.972 = 97.2\%$

Additional questions: Calculate the mean number of defective items in a sample of 20. Answer: $20 \times (0.1) = 2$

2. X is the number of accidents per week.

$$p(10) = P(X = 10) = \frac{e^{-6}6^{10}}{10(9)(8)...1} = 0.0413$$

The standard deviation is $\sqrt{6} = 2.45 = 2$ accidents per week (rounded down).

3. Expected value of $X = \mathrm{E}[X] = \overline{X} = \int_0^2 0.5x\,dx = 0.25x^2\Big|_{x=0}^{x=2} = 1$

Variance of $X = \sigma_X^2 = \overline{X^2} - (\overline{X})^2$
$= \int x^2(0.5)dx\,. - 1$

$= \frac{1}{6}x^3\Big|_{x=0}^{x=2} - 1 = 0.33$

4. Let X denote return (assume all values are annualised).

a) $P(X > 0.20) = P(X > \mu) = 0.5 = 50\%$

b) $P(-0.20 < X < 0.60) = P(\mu - \sigma < X < \mu + \sigma) = 0.68 = 68\%$

Application of the normal distribution to market research

Activity 2.15

Activity

Calculate confidence interval and average

What will you do?

Samples are taken annually to calculate the average number of hours worked per month by agricultural labourers, and their average monthly wages. Standard deviations (std dev) are also calculated.

The data for years 2005–2008 are shown in this table:

	2005	2006	2007	2008
Average monthly hours worked (and std dev)	178 (10)	162 (14)	148 (7.2)	140 (12.1)
Average monthly wages in pounds (and std dev)	370.00 (10.56)	450.20 (5.53)	500.00 (12.54)	480.60 (14.32)
Sample size *n* (number of workers)	200	200	138	112

1. Calculate the 95 per cent confidence interval for each year. Discuss the meaning of the values.
2. Determine the average wages per hour for each year.

Solution

1. The 95% confidence interval is: $(\bar{x} - 1.96s/\sqrt{n},\ \bar{x} + 1.96s/\sqrt{n})$

   Year 2005:
   Confidence interval for hours worked
   $(178 - 1.96\ (10)/\sqrt{200},\ 178 + 1.96\ (10)/\sqrt{200}) = (176.61, 179.39)$

   The average number of hours worked per month in the population of all agricultural labourers lies in this interval with 95% certainty.

   Monthly wages interval
   $(370 - 1.96\ (10.56)/\sqrt{200},\ 370 + 1.96\ (10.56)/\sqrt{200}) = (368.54, 371.46)$

   The average monthly wage in the population of all agricultural labourers lies in this interval with 95% certainty.

   This calculation is repeated for the other years.
2. Average monthly wages per hour for 2005: 370/178 = 2.08 pounds per hour for the sample.

Case study:
Market research

Attempt this case study in market research and hypothesis testing before doing the assessment for this module.

Companies are awarded a special rating for their social commitment, which gives them certain tax benefits. The rating system is based on assessments by the companies themselves as to how much they spend on social projects, improving the living conditions of their workers, combating pollution, and so on. They send an annual report to the Internal Revenue Service (IRS).

The IRS awards the ratings, but it cannot check each company's self-assessment every year. It was initially thought that at least 90 per cent of the self-assessments would be correct and fair. However, the possibility exists of mistakes or outright fraud.

After two years, the IRS has thoroughly checked a sample of the reports of 2,100 companies. It found that 12.01 per cent of reports were not quite correct.

Discuss and test the initial view (hypothesis) of the IRS that at least 90 per cent of reports would be correct.

Solution

Step 1: The null hypothesis H_0 is that at least 90% of reports will be correct. Let us denote the population percentage by л.
H_0: л ≥ 90%

The alternative hypothesis H_1 is that less than 90% of reports will be correct.
H_1: л < 90%

We must choose between these two hypotheses. We either reject H_0 and accept H_1 or we do not reject H_0. It is important to note that not rejecting H_0 does not mean we accept H_0. There is merely not enough evidence to reject it.

Step 2: The actual, but unknown, situation is that H_0 is either true or false. If H_0 is true we may, based on the sample data, reject H_0. In this case, we will have made an error (Type I). In our example, we want the probability of a Type I error to be as small as possible and choose $\alpha = 0.01$.

Step 3: Determine the critical Z values. $Z_\alpha = 2.33$ for $\alpha = 0.01$ and Z_{calc} $(= \dfrac{\bar{x} - \mu}{s/\sqrt{n}})$ is calculated from the sampling distribution. The population mean is л = 90%. The observed mean value was 87.99%. The standard deviation $s/\sqrt{n}$ must be adapted for percentages. It becomes:

becomes: $\sqrt{\frac{\pi(100-\pi)}{n}} = \sqrt{\frac{90(100-90)}{2100}} = 0.655$

and so $Z_{calc} = \frac{87.99 - 90}{0.655} = -3.07$ or $|Z_{calc}| = 3.07$

We see that $|Z_{calc}| > |Z_{\alpha}|$ (3.07 > 2.33)

Step 4: Since $|Z_{calc}| > |Z_{\alpha}|$ we reject H_0. The sample result of 87.99% is 3.07 standard deviations away from the hypothesised population mean of 90%. This is outside the region determined by 2.33 standard deviations. We accept the alternative hypothesis H_1.

Assessment

Assessment — Module 2

1. Find the mean, median and mode of this set of data. Represent the data in a scatter graph:
   15, 6, 14, 7, 12, 11, 7, 10, 8, 13
2. Study the frequency table for the consumption of diesel by a group of trucks. (For example, there are six trucks using between 25 and 30 litres of diesel per 100 km.)

Class: Litres per 100 km	Frequency: Number of trucks
20–25	2
25–30	6
30–35	12
35–40	8
40–45	5

   a) Expand the table to include cumulative and relative frequencies.

   b) Find the mean and median for the distribution.

   c) Draw a histogram and ogive.

   d) Use the ogive to confirm the value of the median.

   e) Is the distribution positively or negatively skewed?

   f) Calculate the standard deviation for the data.

3. Sales figures S_1 and S_2 (income in a local currency) for a week for two branches of a coffee shop are:

Day	Sales for shop 1	Sales for shop 2
1	1,200	1,300
2	2,300	2,450
3	2,400	3,500
4	1,800	2,000
5	1,100	1,500
6	2,000	2,200
7	2,200	1,800

   a) Determine the mean and standard deviation of sales for each shop.

   b) Calculate the co-variance and correlation co-efficient for sales figures S_1 and S_2.

   c) Present the data in column or scatter graphs.

   d) Interpret and discuss your results for 3a) – 3c).

4. What is the probability of getting a sum greater than 10 when throwing two dice?
5. Let E and F be mutually exclusive and independent events. What can you say about the probabilities $P(E \cup F)$ and $P(E \cap F)$?
6. A fair coin is tossed three times. What is the probability of throwing two tails and one head? Use a probability tree for your calculations.
7. The conditional probability $P(E|F)$ is 0.4 and the probability $P(E \cap F)$ is 0.2. What is the value of $P(F)$?
8. Random variable X is binomially distributed. There are 20 trials and the probability of success is 0.15. What is the probability of 10 successes?
9. Random variable Y has the Poisson distribution with mean value 4.2.
   a) What is the variance of the distribution?
   b) What is the probability of 14 successes?
10. Random variable X is normally distributed, with mean value 30 and a variance of 14.
    a) What is the probability that X will take on values greater than 44? (Hint: No tables are necessary.)
    b) What is the probability that X will take on values less than 20? (Hint: Transform X to a standard normal variable Z and use tables.)
11. A company wants to find out how long it takes a worker to assemble a certain component of a machine. A sample of 40 workers shows an average time of 76.4 seconds, and a standard deviation of 17.2 seconds. Find the 95 per cent confidence interval for the population mean.
12. A sample of 500 patients is given a new treatment for AIDS. Testing the patients after one year shows that 67 per cent of these patients have responded very well, with increased CD4- or T-cell counts. Determine a 95 per cent confidence interval for the percentage of patients in the population who will have increased CD4 counts after the treatment.

Solutions

1. Mean: 10.3 Median: 10.5 Mode: 7

2. a)

Class: Litres per 100 km	Frequency *f*: Number of trucks	Cumulative frequency *F*	Relative frequency
20–25	2	2	0.061
25–30	6	8	0.182
30–35	12	20	0.364
35–40	8	28	0.242
40–45	5	33	0.152

b) Mean: Midpoints x_i of classes: 22.5, 27.5, 32.5, 37.5, 42.5

$$\text{Mean} = \frac{1}{N}\sum_{i=1}^{N} x_i f(x_i)$$

$$= \frac{1}{33}[22.5(2) + 27.5(6) + 32.5(12) + 37.5(8) + 42.5(5)]$$

= 33.71 (rounded) litres per 100 km

The middle value is observation 17, which falls in class 30–35. Since 17 – 8 = 9, the middle value takes position 9 of the 12 values in class 30–35. Now 30 + (9/12)5 = 33.75 and this is the median.

c) and d) are straightforward. See Activity 2.5.

e) Skewness is proportional to the difference (mean – median). Skewness is negative here: more than half the values lie to the right of the mean value. The distribution is not symmetrical.

f) Variance $(x) = s^2 = \dfrac{\sum_{i=1}^{N}(x_i - \bar{x})^2 f(x_i)}{\sum_{i=1}^{N} f(x_i)}$ for grouped data values with midpoint values x_i of classes and frequencies $f(x_i)$.

We have $\bar{x} = 33.71$

$s^2 = 29.039$ and $s = 5.389$

3. a) Shop 1: Mean daily sales income $\overline{S1} = 1{,}857.14$

Std dev. = 483.60 (rounded)

Shop 2: Mean daily sales income $\overline{S2} = 2{,}107.14$

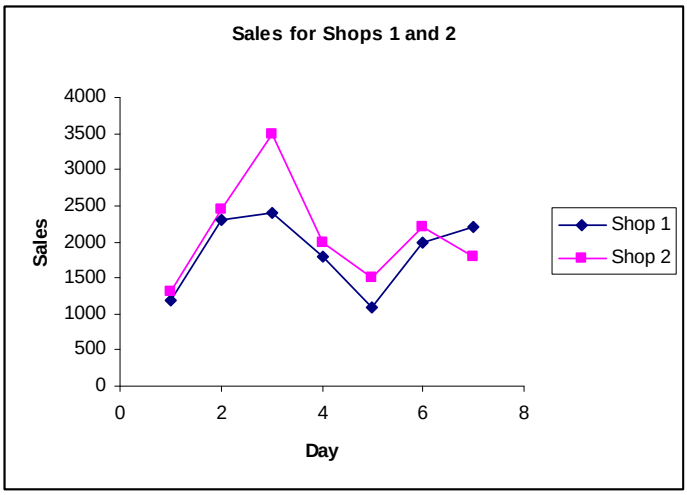
Std dev = 675.29 (rounded)

b) Co-variance = $\dfrac{\sum_{i=1}^{N}(S1_i - \overline{S1})(S2_i - \overline{S2})}{7}$

= 263,265.27

Correlation co-efficient = $\dfrac{263265.27}{(483.6)(675.29)}$ 0.806

c) We show a scatter graph.



d) Shop 2 has a greater average income, but also a greater standard deviation and therefore uncertainty in income. The two incomes from the shops are positively correlated and close to 1: they tend to rise and fall together.

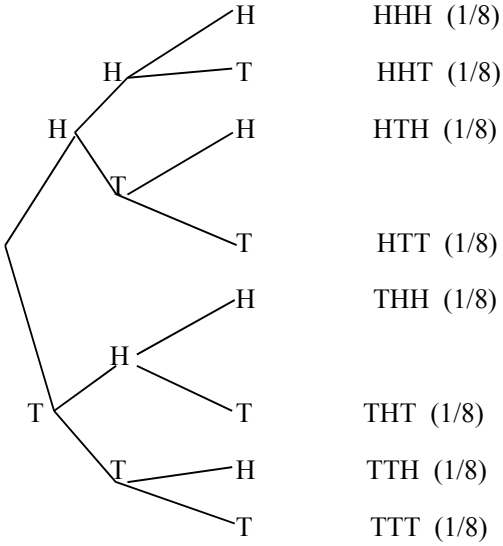
4. There are 36 pairs of outcomes from throwing two dice. Let the sum be random variable S.
$P(S > 10) = P(S = 11) + P(S = 12) = (2/36) + (2/36) = 1/9$

5. $P(E \text{ or } F) = P(E \cup F) = P(E) + P(F)$

$P(E \text{ and } F) = P(E \cap F) = P(E)\,P(F) = 0.$

This means at least one of $P(E)$ or $P(F)$ equals zero, so that at least one of E or F is an impossible event.

6. At the end of each branch we write the string of outcomes and probability. Outcomes with two tails are highlighted:



There are three independent outcomes with two tails and the probability is 3/8.

7. $P(E \mid F) = \dfrac{P(EF)}{P(F)}$

Therefore $P(F) = \dfrac{P(EF)}{P(E|F)} = \dfrac{0.2}{0.4} = 0.5$

8. $P(X = 10) = \binom{20}{10} 0.15^{10}(1 - 0.15)^{20-10}$

$= 0.00021$

9. a) Variance of $X = \sigma^2(X) = Np = \mu = 4.2$

b) $P(X = 14) = \dfrac{e^{-4.2}(4.2)^{14}}{14(13)(12)...1} = 0.00009$

10. a) $P(X > 44) = P(X > \mu+\sigma) = ½(1 - P(\mu-\sigma \leq X \leq \mu+\sigma))$
$= ½(1 - 0.68)$ (from symmetry)
$= 0.16$

b) $P(X < 20) = P(\dfrac{X-\mu}{\sigma} < \dfrac{20-\mu}{\sigma})$
$= P(Z < -2.67) = P(Z > 2.67)$ (symmetry)
$= 0.5 - P(0 < Z < 2.67)$ (see the Appendix)
$= 0.5 - 0.496 = 0.004$

11. A 95% confidence interval around sample mean $\bar{x}$ is:
$(\bar{x} - 1.96s/\sqrt{n}, \bar{x} + 1.96s/\sqrt{n})$

$= (\bar{x} - 1.96s/\sqrt{n},\ \bar{x} + 1.96s/\sqrt{n})$

$= (71.07, 81.17)$

12. A 95% confidence interval around sample mean $\bar{x}$ is:

$$(\bar{x} - 1.96\sqrt{\frac{\pi(1-\pi)}{n}},\ \bar{x} + 1.96\sqrt{\frac{\pi(1-\pi)}{n}})$$

$= (0.67 - 1.96(0.00054), 0.67 + 1.96(0.00054)$

$= (0.669, 0.6711)$

Module 3

Spreadsheet modelling in management

Outcomes

Upon completion of this module students will be able to:

- **master** basic spreadsheet techniques;
- **generate** graphs, charts and tables electronically to display data;
- **apply** sensitivity analysis to cash flow problems;
- **evaluate** projects and capital budgets;
- **use** ANOVA analysis on data;
- **apply** spreadsheet tools such as Data Analysis and Goal Seek; and
- **use** simple macros in VBA

Unit 9

Introduction to spreadsheets

Outcomes

Upon completion of this unit students will be able to:

- **understand** menu bars, toolbars and dropdown menus;
- **enter**, **change** and **delete** data in a cell;
- **use** formulas, references and functions;
- **save** and **retrieve** files;
- **format** the appearance of a cell's contents;
- **change** the width of a column;
- **insert** rows ; and
- **move** data from one location to another.

What is a spreadsheet?

Activity 3.1

Activity

Make a spreadsheet

What will you do?

1. Open a new workbook. Enter this table in cells A1 to B7.

Year	Sales
1	112
2	243
3	265
4	310
5	400
6	450

2. Save the workbook as **Annual Sales Figures**. Change all sales entries by adding 100 to each number. Save the workbook and close.

3. Open the workbook again. Enter the words "Total Sales" in Cell A9.

4. Create a new workbook. Enter this table in cells A1 to B5.

1	3
2	5
3	7
4	9
5	11

5. Copy the table to cells E8 to F12. Save the workbook as **Linear Function**.

Solution

This is a straightforward exercise.

Formulas and functions

Activity 3.4

Activity

Do more calculations

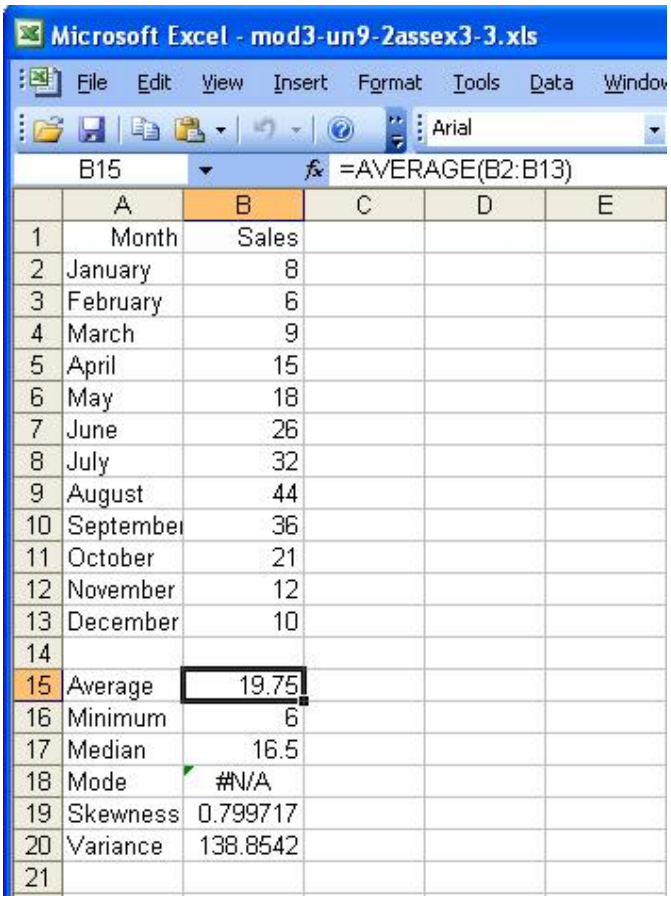
Create a new file in Excel and enter the data from the Module 2 section "Charts, graphs and histograms". The table gives the sales figures for the number of pairs of sunglasses sold in a shop in an island resort.

Month and sales figure		Month and sales figures	
January:	8	July:	32
February:	6	August:	44
March:	9	September:	36
April:	15	October:	21
May:	18	November:	12
June:	26	December:	10

Choose suitable headings and right-align them. Calculate these in the workbook:

1. Average monthly sales figure.
2. Minimum sales figure.
3. Median and mode for sales.
4. Skewness of the distribution of sales.
5. Variance in sales over the year.

Solutions



	A	B	C	D	E
1	Month	Sales			
2	January	8			
3	February	6			
4	March	9			
5	April	15			
6	May	18			
7	June	26			
8	July	32			
9	August	44			
10	September	36			
11	October	21			
12	November	12			
13	December	10			
14					
15	Average	19.75			
16	Minimum	6			
17	Median	16.5			
18	Mode	#N/A			
19	Skewness	0.799717			
20	Variance	138.8542			
21					

Applications: Project evaluation

Activity 3.6

Activity

Determine profitability and analyse sensitivity

What will you do?

Refer to Activity 1.8 in Module 1.

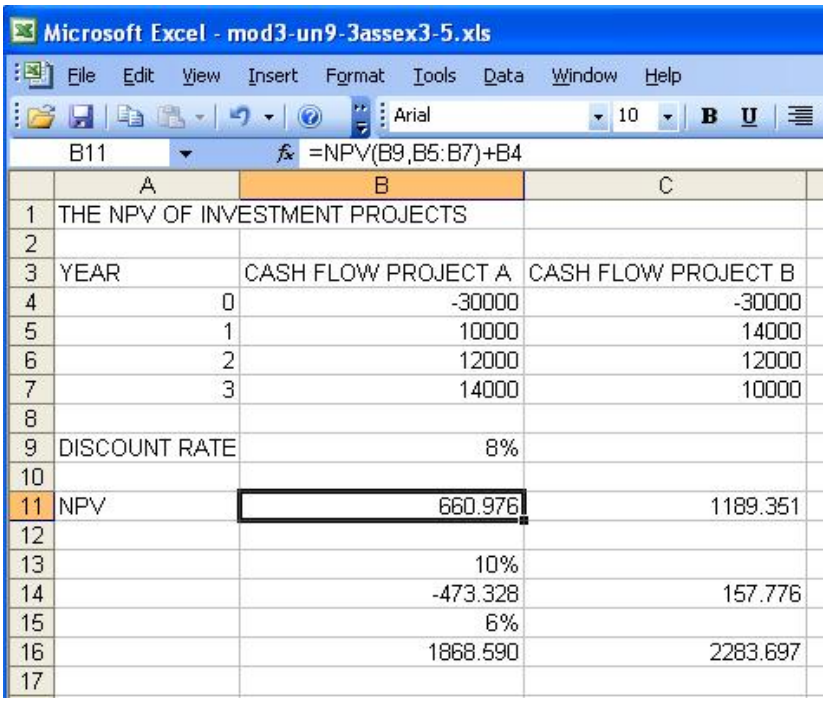
A company in Papua New Guinea has 30,000 PGK (kina or K) to invest in either Project A or Project B. Each project runs over three years. The expected cash inflows at the end of each year are displayed in this table.

Project A	Project B
Year 1: K 10,000	Year 1: K 14,000
Year 2: K 12,000	Year 2: K 12,000
Year 3: K 14,000	Year 3: K 10,000

The required rate of return (discount rate) is 8 per cent p.a.

1. Set up a spread sheet to determine which project is most profitable at this point. Use *NPV* (net present value) as the criterion.
2. Perform a sensitivity ("what if?") analysis to take into account a change in discount rate of 2 per cent, either upwards or downwards.

Solutions



	A	B	C
1	THE NPV OF INVESTMENT PROJECTS		
2			
3	YEAR	CASH FLOW PROJECT A	CASH FLOW PROJECT B
4	0	-30000	-30000
5	1	10000	14000
6	2	12000	12000
7	3	14000	10000
8			
9	DISCOUNT RATE	8%	
10			
11	NPV	660.976	1189.351
12			
13		10%	
14		-473.328	157.776
15		6%	
16		1868.590	2283.697
17			

Unit 10

Tables, charts and graphs in Excel

Outcomes

Upon completion of this unit the student will be able to:

- **set up** tables of data in spreadsheets;
- **generate** frequency tables and histograms;
- **present** data and equations with charts and graphs; and
- **analyse** data with the Descriptive Statistics analysis tool.

Tables and column charts

Activity 3.10

Activity

Get the graphs right

What will you do?

1. The budget for a small farm consists of expenses and income. Total expenses $TE = 10{,}000 + 235N$, where N is the number of units of land planted. Total income $I = 1{,}400N$.

   Determine the approximate budget-balancing (break-even) point with Excel's Chart Wizard. (All numbers are in Ghanaian naira.)

2. The supply and demand for a particular product is described by these functions:
   $P_S = 2Q_S^2 - 6Q_S + 2$ and $P_D = Q_D^2 + 2Q_D + 6$

   (P_S is the price per item the producer will accept and P_D is the price the buyer is willing to pay. Q_S is the number of products supplied and Q_D is the number of products demanded.)

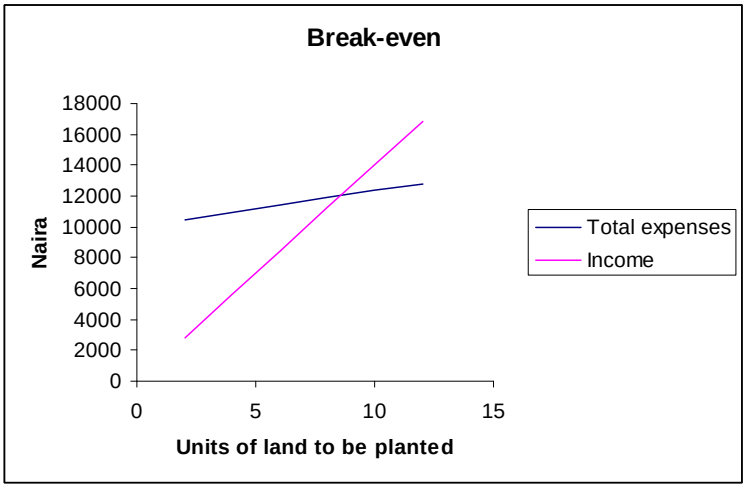
   Use Excel to find the equilibrium point graphically.

Solutions

1.

N	TE	I
2	10470	2800
4	10940	5600
6	11410	8400
8	11880	11200
10	12350	14000
12	12820	16800

Enter data. Highlight range A2:C7 and use Chart Wizard (XY Scatter).



The break-even point can be determined analytically as $N = 8.58$ units of land that need to be planted for costs to equal income.

2. The graphs show portions of the two parabolas.

Analytically, the supply price equals demand price where $Q = 8.45$. The quantity should be rounded off, say to 9 units from the supplier's point of view.

Activity 3.11

Activity

Analyse data

What will you do?

1. Do a Descriptive Statistics data analysis for revenue for the year 1996 in the data for Figure 15.
2. Interpret and discuss each of the values in the Descriptive Statistics box.

Solutions

1.

	A	B
16		
17	*1996*	
18		
19	Mean	147.3333
20	Standard Error	5.936958
21	Median	141
22	Mode	#N/A
23	Standard Deviation	20.56623
24	Sample Variance	422.9697
25	Kurtosis	-0.49284
26	Skewness	0.66683
27	Range	64
28	Minimum	120
29	Maximum	184
30	Sum	1768
31	Count	12

2. The mean monthly company revenue for 1996 was 147.33 (units may be in thousands or millions). Standard error refers to the error if we were doing linear regression on the data.

   The median of 141 means half of the months (that is, 6) had revenue of less than 141, and six months had revenue above 141. There is no mode. The standard deviation shows a deviation around the mean of 20.566 units for the data when seen as a sample.

   The positive skewness means that more than half the values lie to the left of the mean. The negative kurtosis means that tails of the distribution are thinner than for a normal distribution. The probability of extreme values is therefore smaller than for the normal distribution.

Unit 11

ANOVA and Goal Seek

Outcomes

Upon completion of this unit students will be able to:

- **interpret** ANOVA statistics; and
- **find** implicit solutions of equations using Goal Seek.

Comparing data sets with ANOVA

ANOVA stands for **An**alysis **O**f **Va**riance.

Activity 3.14

Activity

Analyse and interpret

What will you do?

The body mass of groups of people from four different countries is measured in kilograms.

Which group has the largest average mass? Can you make a statistically significant judgement?

Use ANOVA analysis and interpret the output.

Group A	Group B	Group C	Group D
67	62	72	58
72	94	65	62
55	110	96	53
58	81	98	57
71	90	118	66

Solutions

Anova: Single factor

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
Group A	5	323	64.6	59.3
Group B	5	437	87.4	311.8
Group C	5	449	89.8	458.2
Group D	5	296	59.2	24.7

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between groups	3651.75	3	1217.25	5.701405	0.007502	3.238872
Within groups	3416	16	213.5			
Total	7067.75	19				

Group C seems to have people with the largest mass and variance in mass.

Consult the F ratio (*F*) and F critical value (*F crit)*.

If F > F crit, then there is a statistically significant difference**.**

If *F* < *F crit,* then the score differences are best seen as chance differences or due to natural variation.

In this case, *F* = 5.7014 and *F crit* = 3.23887. Since clearly *F* > *F crit,* the differences in mass are statistically significant. We can say with certainty (95%) that the masses of these groups show Group C to have individuals with the largest mass.

Goal Seek

Activity 3.15

Activity

Use Goal Seek

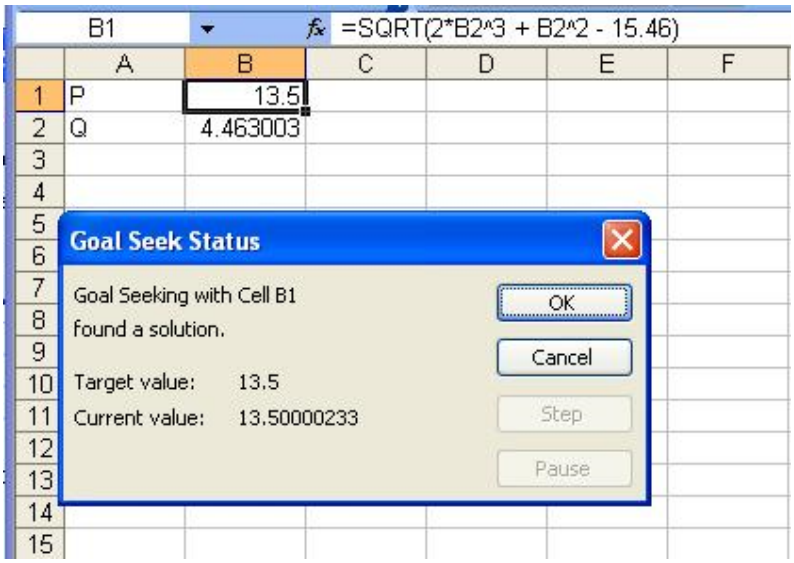
What will you do?

1. The relation between quantities P and Q is given by:
   $P^2 = 2Q^3 + Q^2 - 15.46$
   Use Goal Seek to determine the value of Q when $P = 13.5$.
2. The future value of an ordinary annuity is 24,000 kina.
   There are annual payments of 1,500 kina every year for 10 years.
   What was the annual rate of return over the period?
   (See Module 1, Unit 1, "Annuities" and use Goal Seek.)

3. Find the monthly interest rate R if the PV of four cash inflows C_1, C_2 , C_3 and C_4 at the end of each of 4 months is given as: $PV = 600$, $C_1 = 150$, $C_2 = 200$, $C_3 = 250$, $C_4 = 280$

Solutions

1.



The value of Q is 4.463.

2. $FV = R\dfrac{(1+i)^n - 1}{i}$ with $R = 1{,}500$, $n = 10$, $FV = 24{,}000$

Value for i is 0.1008 = 10.08%

3. $PV = \dfrac{C_1}{(1+r)} + \dfrac{C_2}{(1+r)^2} + \dfrac{C_3}{(1+r)^3} + \dfrac{C_4}{(1+r)^4}$

C1	150
C2	200
C3	250
C4	280
PV	600.0003
Rate	0.154539

$R = 15.45\%$.

Unit 12

Other spreadsheet facilities for model building

Outcomes

The section on VBA can be considered optional. However, VBA is used extensively in the management and financial environment.

Upon completion of this unit students will be able to:

- **apply** a range of Excel functions in building management and business models;
- **implement** simple VBA; and
- **understand** the use of macros.

More Excel functions

Activity 3.17

Activity

Practise these functions

What will you do?

Figure 24 shows an extract from a data set of daily share price returns (%) for five shares over the first 132 trading days in 2008. The total data set is in a spreadsheet.

Daily return (%)

Trading day no.	Share A	Share B	Share C	Share D	Share E
123	2.0	3.55	−4.5	3.4	1.48
124	−1.2	2.1	0.0	1.2	0.0
125	0.0	−2.4	−0.8	1.05	0.0
126	1.4	−3.0	0.6	0.0	−2.8
127	3.6	1.0	1.0	1.18	−1.73

Figure 3 Extract from data set

All share price returns are normally distributed.

Copy the extract into a spread sheet in the range A1:F6.

1. It has been estimated that for 2008 the average daily return for Share B is 1.75 per cent and the daily volatility is 2.52 per cent.

   What is the probability that the daily return for Share B will be less than −1 per cent? (Use NORMDIST. Also solve this manually using the Appendix at the end of Module 2.)

2. You want to know if the daily return for Share D was higher than 1% on day 124. If that was the case, let "GOOD" appear in cell G3; otherwise let "BAD" appear. (Use the "IF" function).

3. Find the return for Share C for day 126 using the VLOOKUP function.

4. Generate random percentage returns for 20 days for Share B. (You must use the normal distribution from the RNG function.)

Solutions

1. From the Appendix:

$$P(R_B < -0.01) = P\left(\frac{R_B - 0.0175}{0.0252} < \frac{-0.01 - 0.0175}{0.0252}\right)$$

$$= P(Z_B < -1.091)$$

$$= 0.5 - 0.36214$$

$$= 1.3786$$

   Using NORMDIST:
   Enter =NORMDIST(-0.01,0.0175,0.0252,TRUE)

   Answer = 0.1375

2. Enter =IF(E17>0.01,"GOOD","BAD") in any cell. In this case, we had the value of D for Day 124 in cell E17.

   The value "GOOD" will appear.

3. =VLOOKUP(126,A15:F20,5,FALSE). The value 0 is returned.

4. Returns will vary.

Case study:
Group research project

This case study involves accountability ratings for companies. Accountability is defined as "a company's ability to explain and justify its actions, and to take responsibility for the consequences of those actions" (Accountability Rating™).

Accountability Rating™ evaluates companies in four key areas (see www.accountabilityrating.com):

1. Strategic intent: Does the company seek to address important social, environmental and broader economic issues in its core business strategy?

2. Governance and management: Are senior executives and the board accountable to stakeholders when setting strategy and formulating policy on extra-financial issues?
3. Engagement: Does the company engage in dialogue with the people and groups who have an interest in its business, may be affected by it or have an effect on it? Does the company publicly report its social and environmental performance?
4. Operational performance: How effective has the company been in implementing its management systems and engagement mechanisms?

The four areas are equally weighted and companies are scored. Accountability Rating™ publishes the scores for the world's largest 100 companies in *Fortune* magazine every year. It also ranks countries and regions.

A study was done in South Africa using Accountability Rating™ measures ("Correlates of corporate accountability among South Africa's largest listed companies by Eccles, Pillay & de Jongh, *Southern African Business Review*, 2009). The authors reported statistically significant relationships between accountability and company size and industry sector. On this basis, they suggest that a company's size and its industry sector motivate socially responsible corporate behaviour.

Group assignment: Your group should do at least one of these:

1. Register on www.accountabilityrating.com and study the methodology of the rating model.

   Study the findings. Write a paper discussing and/or critiquing the methodology and findings.

2. Read the paper "Correlates of corporate accountability among South Africa's largest listed companies". You can access the paper from the journal's website: http://www.unisa.ac.za/sabusinessreview.

   Do a data analysis on the authors' data to see whether you agree with their findings. Write a report.

This is a more challenging and long-term project and can be tackled at the end of Module 5: set up a questionnaire along the lines of Accountability Rating™ and send it to companies in your own country. Collect and organise the data and perform a data analysis like the one undertaken by the *Southern African Business Review*. Write a short paper discussing your results and see whether you can have it published in a local journal or magazine.

Solutions

It is the Lecturer's decision whether the student's work is evaluated.

Assessment

Assessment — Module 3

1.

a) Do a Descriptive Statistics Analysis for the data in Figure 34. Discuss all results.

Year	Pension fund return %
2000	10.00
2001	11.50
2002	13.80
2003	14.43
2004	10.50
2005	15.00
2006	12.28
2007	17.45
2008	−1.06
2009	−2.00

Figure 4

b) Present the data in a column chart and record a macro in VBA as you draw the chart.

2. You need a frequency distribution for the table of data. It describes the consumption of electricity (in kilowatt hours) of 40 households over the period of a month. Present the outcome as a histogram.

122	88	145	150	79
143	145	134	157	145
210	163	234	85	156
88	110	162	99	154
122	156	210	245	160
145	184	215	141	183
163	215	163	107	145
143	163	145	200	156

Figure 5

3.

a) Find the value of the internal rate of return *IRR* from the relation:

$$\frac{C_1}{(1+IRR)} + \frac{C_2}{(1+IRR)^2} + \frac{C_3}{(1+IRR)^3} + \ldots + \frac{C_n}{(1+IRR)^n} = C_0$$

The annual cash flows are:

$C_0 = 12{,}000$, $C_1 = 5{,}000$, $C_2 = 6{,}000$, $C_3 = 6{,}500$, $C_4 = 5{,}000$, $C_5 = 6{,}000$

b) Use both Goal Seek and the *IRR* function to find and compare answers.

c) Investigate the changes in *IRR* as you vary the initial investment C_0.

4. The "intelligence quotients" (IQ) of three groups of students are tested. Groups A, B and C correspond to highest-income families to lowest-income families. The researchers claim that students in Group A are the most intelligent because their average IQ is above 120 (namely 123.75). They also claim the students in Group C will not finish college because their average IQ is below 120, and the lowest at 117.88.

a) On a purely statistical basis, analyse the results (using ANOVA) and report your findings.

b) Critique the study on grounds such as sample size, ethics and the validity of using a measure such as IQ.

The data collected by the researchers is displayed in Figure 36:

Group A	Group B	Group C
135	120	120
130	125	118
100	110	125
140	130	128
100	90	125
95	140	120
145	100	130
105	118	120

Figure 6

5. Write a VBA program for the data in Question 4 that enables a user to find out how many students had IQ scores above a certain value in each group.

6. Figure 37 gives an extract from a data set of daily share price returns (%) for four shares during 2007.

Daily return (%)

Trading day no.	Share 1	Share 2	Share 3	Share 4
101	1.0	1.5	−4.5	2.4
102	−1.2	2.1	0.1	−1.2
103	0.5	−1.4	1.8	1.05
104	2.4	−0.4	0.6	0.0
105	1.6	−1.0	1.2	−1.18
106	0.4	0.2	1.0	−2.2

Figure 7

All share price returns are normally distributed.

Copy the extract into a spread sheet in the range A1:E7.

a) It has been estimated that, for 2007, the average daily return for Share 1 is 2.1 per cent and the daily volatility is 3.52 per cent. What is the probability that the daily return for Share 1 will be more than 2.5 per cent? (Use NORMDIST. Also solve this by hand, using the table at the end of Module 2).

b) You want to know if the daily return for Share 2 was higher than 2 per cent on day 102. If that was the case, let “HIGH” appear in cell F3; otherwise let “LOW” appear.

c) Find the return for Share 3 for day 105 using the VLOOKUP function.

d) Generate random percentage returns for 20 days for Share 4. (Use the normal distribution from RNG as random number generator.)

Solutions

1. a) and b)

Return	
Mean	10.19
Standard error	2.076933
Median	11.89
Mode	#N/A
Standard Deviation	6.567837
Sample variance	43.13649
Kurtosis	0.578499
Skewness	-1.27784
Range	19.45
Minimum	-2
Maximum	17.45
Sum	101.9
Count	10

The mean return is 10.19% and the standard deviation (volatility) is 6.57%. The median is 11.89%.

The negative skewness means that more than half the values lie to the right of the mean. The positive kurtosis means that tails of the distribution are fatter than for a normal distribution. The probability of extreme values is therefore bigger than for the normal distribution.

2. Enter data in range A1:E8. This is the data array. Then obtain

MIN 79
MAX 245
Select intervals: 0-80, 81-160, 161-240, 241-320

Enter endpoints in cells, say A14:A17. This is the bin array.

Highlight cells B14:B17 and do =FREQUENCY(data array, bin array). The result is:

80	1
160	25
240	13
320	1

Or do Data Analysis, Histogram to obtain frequency and histogram.

Lecturers to review graph for correctness.

3. a) IRR = 37.17% (both Goal Seek and function IRR). If an answer in Excel is not in the right format in terms of number of decimal places or notation, right-click on the cell containing the number and click <Format Cells>. Make the necessary changes.

b) and c) The bigger initial investment C_0, the smaller the IRR for the same cash inflows in years 1 to 5.

4. a) Use ANOVA.

Anova: Single factor

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
Group A	8	990	123.75	319.6429
Group B	8	958	119.75	136.2143
Group C	8	943	117.875	27.55357

ANOVA

Source of variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between groups	144.0833	2	72.04167	0.447084	0.645437	3.4668
Within groups	3383.875	21	161.1369			
Total	3527.958	23				

Group C has the smallest IQ average, but the results are not statistically significant. It is clear that the variation within groups is larger than the variation between groups. $F < F_{crit}$ shows that the score differences are best seen as chance differences.

b) The sample size is so small as to have almost no value. Opinions about the value of a measure such as IQ differ. In any case, it is not ethical to use a quantitative measure indiscriminately to make judgements about the future contributions and values of humans.

5. Use the example for the macro Sub CountHighSales() in the Course Manual.

```
Sub CountHighIQ()

  Dim i As Integer

  Dim j As Integer

  Dim numberHigh As Integer

  Dim scoreCutoff As Integer

 scoreCutoff = InputBox("What score value do you want to check for?")

  For j = 1 To 3

    numberHigh = 0

    For i = 1 To 8

      If Range("A2:C9").Cells(i, j) >= scoreCutoff Then _
```

```
            numberHigh = numberHigh + 1
        Next i
    MsgBox "For Group" & j & ", scores were above" & _
        Format(scoreCutoff, "0 000") _
            & "on" & numberHigh & "of the eight students."
    Next j
End Sub
```

6. a) Enter =NORMDIST(0.025,0.021,0.0352,TRUE)

   Value returned: 0.545237

   Therefore $P(S_1 > 0.025) = 54.52\%$

   b) Enter =IF(C3>0.02,"HIGH","LOW") in cell F3. The value "HIGH" will appear.

   c) Click on a cell and enter =VLOOKUP(105,A2:E7,4,FALSE). The value 1.2 is returned.

The worksheet can look like this:

Trading day no.	Share 1	Share 2	Share 3	Share 4
101	1	1.5	−4.5	2.4
102	−1.2	2.1	0.1	−1.2
103	0.5	−1.4	1.8	1.05
104	2.4	−0.4	0.6	0
105	1.6	−1.0	1.2	−1.18
106	0.4	0.2	1	−2.2

HIGH

1.2

0.545237

   d) All answers will, of course, be different.

Module 4

Modelling in management: Optimisation and forecasting

In this module we address the advantages and possible dangers of modelling; the use of Markov processes; simulation, linear and non-linear programming; forecasting in decision making; and time series and linear regression. We will also look at applications in Excel.

Upon completion of this module students will be able to:

Outcomes

- **understand** the philosophy, methods and risks of modelling;
- **apply** modelling to stochastic processes;
- **solve** optimisation problems with linear, integer and non-linear programming techniques;
- **use** simulation and forecasting methods, including regression, to inform decisions; and
- **implement** modelling tools in Excel.

Unit 13

Optimisation and forecasting

Introduction to modelling and decision making

Outcomes

Upon completion of this unit students will be able to:

- **describe** the steps in the modelling process;
- **explain** the importance and limitations of models;
- **distinguish** between modelling assumptions and reality; and
- **apply** Markov chains to the modelling of stochastic processes.

Advise students of possible resources (if necessary). Research and reports can be done by groups.

The modelling process and model risk

Case study:
Model risk and crises

Choose one of the historical financial crises mentioned in this unit. Research it and write a short report. If possible, mention the positive and/or negative roles that quantitative models played or could have played in the crisis.

Solutions

It is the Lecturer's decision whether the student's work is evaluated.

Models and assumptions

Activity 4.1

Activity

Consider random walk

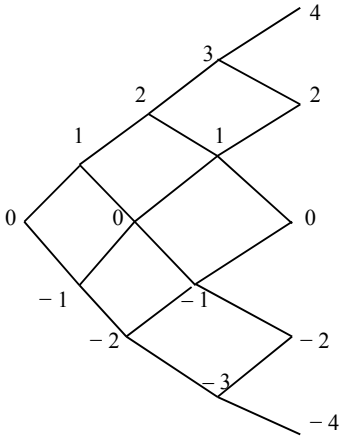
1. Consider the random walk of Example 2: suppose you are standing at point $X_0 = 0$. A coin is flipped every second and if it is heads you take one step forward and if it is tails you take one step backwards. Your position at any time t is a random variable X_t.
   a) Construct a binomial tree with four time steps for X_t.
   b) Show that the expected value $E[X_t] = 0$ for each t.
   c) Prove that $var[X_t] = t$.

d) Show that at each node on the tree the value of X_t equals the expected value of the two values X_{t+1} following from that node. Use this fact to reason that if the values of X at the end nodes are given, then you can work backwards to find the value X_0 at time $t = 0$.

2. Explain the link between the efficient market hypothesis and the Markov model.

Solutions

1. a) The process is a simple random walk.



t=0	t=1	t=2	t=3	t=4
X_0	X_1	X_2	X_3	X_4

b) $E[X_3] = \frac{1}{8}(X_0 + 3) + (3 \times \frac{1}{8})(X_0 + 1) + (3 \times \frac{1}{8})(X_0 - 1) + \frac{1}{8}(X_0 - 3)$
$= X_0$
$= 0$

Probabilities are the product of individual probabilities of each individual independent step.

(Note that there are three paths each leading to positions 1 and −1 at $t = 3$.)

This proof can be repeated for each time t.

c) At t = *2*:

$\text{var}[X_2] = E[X_2^2] - (E[X_2])^2$
$= (2)^2 (\frac{1}{4}) + (0)^2 (2 \times \frac{1}{4}) + (-2)^2 (\frac{1}{4})$
$= 1 + 1$
$= 2$
$= t$

This proof can be repeated for each time t.

d) Consider final nodes $X_4 = 4$ and $X_4 = 2$, which follow from $X_3 = 3$: the probabilities of reaching positions $X_4 = 4$ or $X_4 = 3$ *given* that we are at position $X_3 = 3$ are ½ each. The expected value we are computing is a *conditional* expectation.

$$E[X_4 = 4 \text{ and } X_4 = 2 \mid X_3 = 3] = 4(½) + 2(½)$$
$$= 3$$
$$= X_3$$

Properties a–d are the properties of random walk or Wiener process X_t. This stochastic process X_t is called a martingale.

Moving backwards from pairs of values in this way brings us to the value at $t = 0$. This allows us to find the price of products such as options from their final values.

2. Briefly, our possible positions at any time t are fully determined by our information of our position at the previous time $t - 1$. Information about our position at $t - 2$, for example, does not add any additional knowledge. This is one way of looking at the efficient market hypothesis.

Unit 14

Simulation

Outcomes

Upon completion of this unit students will be able to:

- **describe** the steps in running a simulation;
- **generate** random numbers; and
- **understand** how to analyse uncertain problems using simulation.

Simulation imitates the behaviour of stochastic situations. Monte Carlo simulation uses random number generation to simulate possible outcomes.

Activities can be constructed along the lines of Activity 4.3. It is not possible to give specific answers to such problems, since random numbers are, well, random.

Unit 15

Linear and non-linear programming

Outcomes

Upon completion of this unit students will be able to:

- **calculate** the optimal value of quantities under constraints using linear programming methods;
- **understand** the basics of integer programming;
- **determine** maxima or minima with non-linear programming; and
- **apply** the methods using Excel.

Introduction to linear programming (LP)

Activity 4.6

Activity

Find the maximum

What will you do?

1. Find the maximum of the function F in Activity 4.5 using the corner-point method.
2. Find the minimum of the function F in Activity 4.5 using the graphic and corner-point methods.
3. Find the maximum of function $G(x, y) = 4x + 2y + 8$ subject to the constraints $0 \le x \le 3$, $y \ge 0$ and $y - x \le 1$.
4. Find the minimum of function $F(x, y) = x - 2y + 6$ subject to the constraints $0 \le x \le 3$ and $0 \le y \le 3$.

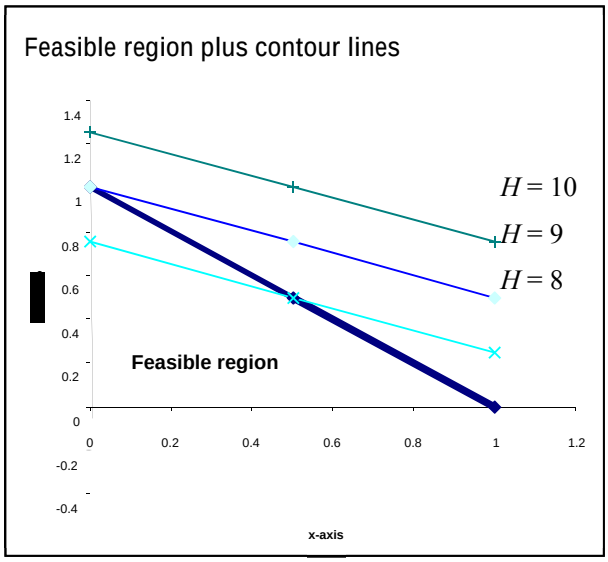
Solutions

1. $F(x, y) = 2x + 4y + 5$. The units of F are in thousands of dollars. Variables x and y represent quantities of two items A and B that are sold, and their units are in thousands. The constraints are $x \ge 0$, $y \ge 0$ and $x + y \le 1$. (Quantities of items sold must be non-negative and we cannot sell more than 1,000 items altogether.)

   Corner-point method: the feasible region has corner points (0, 0), (1, 0) and (0, 1). $F(0, 0) = 2(0) + 4(0) + 5 = 5$
   $F(1, 0) = 2(1) + 4(0) + 5 = 7$
   $F(0, 1) = 2(0) + 4(1) + 5 = 9$

   The maximum is at $(x, y) = (0, 1)$ and the value is 9.

2.



2. From the graph, it is clear that the contour line ($y = -0.5\,x + \frac{H-5}{4}$) which touches the feasible region at (0, 0) gives the minimum of F.
Min $F = F(0, 0) = 5$

3. $G(x, y) = 4x + 2y + 8$ subject to the constraints $0 \le x \le 3$, $y \ge 0$ and and $y - x \le 1$.

   Corner points of feasible region: (0, 0), (3, 0), (3, 4) and (0, 1).
   $G(0, 0) = 8$
   $G(3, 0) = 20$
   $G(3, 4) = 28$
   $G(0, 1) = 10$

   The maximum is at $(x, y) = (3, 4)$ and the value is 28.

4. Minimum of function $F(x, y) = x - 2y + 6$ subject to the constraints: $0 \le x \le 3$ and $0 \le y \le 3$

   Corner points of feasible region: (0, 0), (3, 0), (3, 3) and (0, 3).
   $F(0, 0) = 6$
   $F(3, 0) = 9$
   $F(3, 3) = 3$
   $F(0, 3) = 0$

   The minimum is at $(x, y) = (0, 3)$ and the value is 0.

Introduction to non-linear programming

Activity 4.7

Activity

Maximise and minimise

What will you do?

Classify the problems as LP, ILP or NLP. Draw all feasible regions and a level line for the object function where possible.

1. Maximise $F(x, y) = x + y$ subject to the constraints $0 \leq x \leq 3$, $y \geq 0$ and $y^2 + x^2 \leq 1$.
2. Minimise $G(x, y) = 2x - 3y$ subject to the constraints $x, y \geq 0$ and $y + x \leq 1$.
3. Maximise $F(x, y) = x\,y + 1$ subject to the constraints $0 \leq x \leq 3$ and $0 \leq y \leq 1$.
4. Maximise $K(x, y) = x + y$ subject to the constraints $0 \leq x \leq 3$ $y \geq 0$, $y + x \leq 3$ and x and y both integers.

Solutions

1. NLP. Feasible region is the quarter circle segment of radius 1 in the first quadrant. Level lines have the equation $y = H - x$ for varying H.
2. LP. Feasible region is the triangle in the first quadrant with corner points (0, 0), (0, 1), (1, 0). Level lines have the equation $y = (2/3)x - (1/3)H$ for varying H.
3. NLP. Feasible region is the rectangle in the first quadrant with corner points (0, 0), (3, 0), (3, 1) and (0, 1). Level lines have the equation $y = (H - 1)/x$ for varying H.
4. ILP. Feasible region consists of the 10 integer pairs (x, y) inside the triangle in the first quadrant with corner points (0, 0), (0, 3) and (3, 0). Level lines have the equation $y = H - x$ for varying H.

Lecturers should review graphs for correctness.

Applications in Excel

The Solver tool for linear programming

Activity 4.8

Activity

Minimise the cost

What will you do?

The cost C associated with manufacturing cables depends on the cost of copper wire and plastic tubing. Let x denote the number of metres of copper wire and y the number of metres of plastic tubing. There are fixed costs of INR 200,000 before manufacturing even starts. The cost of copper wire is INR 200 per metre and the cost of plastic tubing is INR 5.45 per metre. Material is only delivered if you order a combined length of copper and plastic that is more than 500 m. On the other hand, restrictions on storage space mean the combined length of copper and plastic can be 1,000 m at most. As well, operationally you should always have at least 200 m of copper wire and 100 m of plastic tubing at the factory.

Use Solver to find the numbers of metres of copper wire and of plastic tubing that will minimise the cost.

Solution

The model is:

Minimise cost $C(x, y) = 200{,}000 + 200x + 5.45y$

Subject to constraints $500 \leq x + y \leq 1{,}000$

$$x \geq 200$$

$$y \geq 100$$

Create a workbook in Excel. Enter:

Cell A1: "Cost F" Cell B1: =(200000 + 200*B2+5.45*B3)

Cell A2: "x" Cell B2: 0

Cell A3: "y" Cell B3: 0

Cell A4: "$x+y$" Cell B4: =(B2+B3)

The values of 0 in B2 and B3 are "starter" values for x and y and you could enter any reasonable values. We need these cells because they contain the variables for which we are optimising F. They also appear in the constraints. We need cell B4 because "$x+y$" appears in a constraint.

Cost F	200000
x	0
y	0
x+y	0

Click on B1. Now click <Tools> and then Solver. (To see that the Solver Add-in has been activated, click <Tools>, <Add-ins> and then ensure that Solver Add-In has been ticked.)

A box with Solver Parameters will appear. $B1$ should be in "Set Target Cell". Remember that the $ symbol on either side of a cell name means that the cell is an absolute reference and will not change.

In the next line click "Min" because we want to minimise *F*. In "By changing cells" enter B2,B3.

Click in "Subject to Constraints" box and then click on <Add>. The "Add Constraint" box pops up. In "Cell Reference" enter B2, in the next box choose inequality >= and in the final box enter 200. Click <OK>. You have entered constraint $x \geq 200$. Click <Add>.

The "Add Constraint" box pops up. In "Cell Reference" enter B3, in the next box choose inequality >= and in the final box enter 100. Click <OK>. You have entered constraint $y \geq 100$. Click <Add>.

The "Add Constraint" box pops up. In "Cell Reference" enter B4, in the next box choose inequality <= and in the final box enter 1. Click <OK>. You have entered constraint $x + y \leq 1000$. Click <Add>.

The "Add Constraint" box pops up. In "Cell Reference" enter B4, in the next box choose inequality >= and in the final box enter 500. Click <OK>. You have entered constraint $x + y \geq 500$.

Now click <Solve>. The following box appears.

Cost F	241635
x	200
y	300
x+y	500

The cost is minimised by keeping 200 m of copper and 300 m of plastic for manufacturing. The minimum cost is 241,635 rupees.

The Solver tool for non-linear programming

Activity 4.9

Activity

Solve an ILP problem

What will you do?

1. Use Excel to solve this ILP problem.

   Maximise $(2x + 3y - 5z)$ subject to:

   $x + y \leq 12$, $x + y + z \leq 20$, $x \leq 10$, x, y and z positive integers.

2. In the portfolio problem above, why can you not use these starter values: a: $x_1 = 1$, $x_2 = 1$ and $x_3 = 0$?

   Change the co-variances to positive values:

   $\sigma_{X1,X2} = 0.025$, $\sigma_{X1,X3} = 0.044$ and $\sigma_{X2,X3} = 0.0542$

   and repeat the optimisation problem.

   Discuss your findings.

Solutions

1. Do as for LP, with additional constraints for positivity and integer values.

   Click <Add> in the "Add Constraint" box. In "Cell Reference" enter B2, and in the next box choose "int" and click <OK>. You have entered constraint "x is an integer". Click <Add> again and in "Cell Reference" enter B3, in the next box choose "int" and click <OK>. You have entered constraint "y is an integer". And so on. The result is:

F	36
x	0
y	12
z	0
x+y	12
x+y+z	12

2. These seed values do not satisfy constraint $x_1 + x_2 + x_3 = 1$.

   We consider a portfolio P of three shares, X1, X2 and X3. It is known that their average annual rates of return over the last five years were:

   $E(R_{X1}) = 7.6\%$, $E(R_{X2}) = 14.4\%$ and $E(R_{X3}) = 16.9\%$

   Their annual volatilities are estimated as:
   $\sigma_{X1} = 25.1\%$, $\sigma_{X2} = 45.25\%$ and $\sigma_{X3} = 49.2\%$

   The new co-variances are:

   $\sigma_{X1,X2} = 0.025$, $\sigma_{X1,X3} = 0.044$ and $\sigma_{X2,X3} = 0.0542$

The problem is to determine the proportions x_1, x_2 and x_3 of the investor's money that should be put in shares X1, X2 and X3, respectively. The investor wants to minimise her risk while requiring a rate of return of at least 12% on portfolio P. No short-selling is allowed.

The output is:

E(RP)	0.12
vol(RP)	0.267238
x1	0.448883
x2	0.290155
x3	0.260962
Sum(xi)	1

Compare this to the output for the original problem with the negative co-variances:

E(RP)	0.12
vol(RP)	0.201919
x1	0.434607
x2	0.343263
x3	0.22213
Sum(xi)	1

The volatility for the portfolio with positive co-variances only is larger than for the original problem with some negative co-variances. This means more risk.

Case study: Employee work schedules

A factory operates seven days a week, 24 hours a day , with workers doing shifts. On certain days, more workers are needed to finish off and load products for shipping. The manager estimates the minimum numbers of workers needed on different days are:

Mon	Tue	Wed	Thu	Fri	Sat	Sun
18	12	20	12	22	20	8

On Mondays at least 18 workers are needed; on Tuesdays at least 12 workers, and so on. The shifts are 24 hours long but designed to let workers rest in between. Certain shifts are better paid because they extend late at night or are on Saturdays. There are four kinds of shift, described as:

Shift type	Days on duty	Weekly wage
1	Monday, Wednesday, Saturday	USD 560
2	Tuesday, Friday, Sunday	USD 570
3	Monday, Thursday, Saturday	USD 550
4	Wednesday, Friday	USD 300

Problem: The manager wants to minimise wage expenses.

How many workers should the manager assign to each shift so that the minimum number of workers will be available each day as required?

Hint: Assign variables as follows:

x_1 = number of workers assigned to Shift 1, x_2 = number of workers assigned to Shift 2, x_3 = number of workers assigned to Shift 3 and x_4 = number of workers assigned to Shift 4.

Formulate the object function as the total weekly wage expense. Then formulate the seven constraints and use Solver in Excel. Remember, this is an ILP problem. To help you, here is one constraint: since you must have at least 18 workers on Mondays, you must have $x_1 + x_3 \geq 18$.

Solutions

Total weekly wages function:
$F(x_1, x_2, x_3) = 560\, x_1 + 570\, x_2 + 550\, x_3 + 300\, x_4$

Constraints: $x_1 + x_3 \geq 18$, $x_2 \geq 12$, $x_1 + x_4 \geq 20$, $x_3 \geq 12$,$x_2 + x_4 \geq 22$,
$x_1 + x_3 \geq 20$,$x_2 \geq 8$

$x_1, x_2, x_3, x_4 \geq 0$

x_1, x_2, x_3, x_4 are integers.

The constraints can be reduced: for example, the two constraints $x_1 + x_3 \geq 18$ and $x_1 + x_3 \geq 20$ reduce to one constraint: $x_1 + x_3 \geq 20$.

Use Solver. The output is:

Objective function F	21520
x1	8
x2	12
x3	12
x4	12
x1+x3	20
x1+x4	20
x2+x4	24

This gives the number of workers per shift as 8, 12, 12, 12. The minimum total wage cost per week is USD 21,520. (The question of whether cost is the only determinant for assigning workers to shifts is another matter.)

Unit 16

Forecasting: Time series and linear regression

Outcomes

Upon completion of this unit students will be able to:

- **describe** different methods for forecasting in management;
- **calculate** trends with moving averages;
- **apply** forecasting with time series;
- **apply** basic linear regression in forecasting; and
- **explain** the limitations of forecasting methods.

Moving averages and exponential smoothing

Activity 4.12

Activity

Forecast

What will you do?

1. Consider the table of observed values y_t for six monthly time steps t.

t	1	2	3	4	5	6
y_t	9.4	10	11.02	10.8	11	11.3

   a) Use the simple averages method to find a forecast for F_7.

   b) Use the two-month moving averages method to find a forecast for F_7.

   c) Use the four-month moving averages method to find a forecast for F_7.

   d) Apply exponential smoothing with $\alpha = 0.2$ to find a forecast for F_7. Let $F_1 = 9$.

2. Consider the table of observed and forecast values.

t	1	2		4
y_t	2.1	2.3	2.6	2.4
F_t	2	2.4	2.5	2.6

   Determine the ME, MAD and MSE.

Solutions

1. a) $F_7 = \dfrac{\sum_{t=1}^{6} y_t}{6} = 10.59$

   b) $F_7 = \dfrac{\sum_{t=5}^{6} y_t}{2} = 11.15$

   c) $F_7 = \dfrac{\sum_{t=3}^{6} y_t}{4} = 11.03$

   d) $F_7 = \sum_{k=1}^{6} 0.2(0.8)^{k-1} y_{7-k} + (0.8)^6 (9)$ (with $\alpha = 0.2$)

   $= 0.2(11.3) + 0.2(0.8)(11) + 0.2(0.8)^2(10.8) + 0.2(0.8)^3(11.02)$
   $+ 0.2(0.8)^4(10) + 0.2(0.8)^5(9.4) + (0.8)^6(9)$

   $= 7.966 + 2.359$

   $= 10.325$

2. ME $= \dfrac{\sum_{t=1}^{4} e_t}{4}$ where $e_t = y_t - F_t$

MAD $= \dfrac{\sum_{t=1}^{4} |e_t|}{4}$

MSE $= \dfrac{\sum_{t=1}^{4} (e_t)^2}{4}$

Determine errors: $e_1 = 0.1$, $e_2 = -0.1$, $e_3 = 0.1$, $e_4 = -0.2$

$|e_1| = 0.1, |e_2| = 0.1, |e_3| = 0.1 \ |e_4| = 0.2$

$(e_1)^2 = 0.01, (e_2)^2 = 0.01, (e_3)^2 = 0.01, (e_4)^2 = 0.04$

ME $= -0.025$, MAD $= 0.125$, MSE $= 0.0175$

Time series models

The method of simple linear regression

Activity 4.14

Activity

Consider sales data

What will you do?

Consider data (t, y_t), $t = 1, 2, 3$ for sales in your shop. Sales are in thousands of items.

Year (t)	Sales (y_t)
1	10
2	8
3	5

1. Use linear regression to find the equation of the line of best fit for the data in the table.
2. Draw the graph of the regression model for sales.
3. Determine the value of one of the measures of goodness of fit. Discuss its meaning.
4. Forecast the values of y at $t = 4$, and $t = 4.5$
5. After how many months will you have no sales?

Solutions

1. Slope $m = \dfrac{3\sum ty_t - \sum t \sum y_t}{3\sum t^2 - (\sum t)^2}$

$$= \frac{3(10+16+15) - 6(23)}{3(14) - 6^2}$$

$$= -2.5$$

y-intercept $c = \bar{y} - m\bar{t}$

$= (23/3) - (-2.5)2$

$= 12.67$

Straight line of best fit: $y = -2.5t + 12.67$

2. The graph is the straight line with y-intercept of 12.67 and negative slope −2.5, and can be drawn by hand or in Excel. (See also the additional section below Activity 4.14.)

   Lecturers should review graphs for correctness.

3. The co-efficient of determination, r^2, is such a measure.

$$r^2 = \frac{[3\sum ty_t - \sum t \sum y_t]^2}{[3\sum t^2 - (\sum t)^2] \times [3\sum y_t{}^2 - (\sum y_t)^2]}$$

$$= \frac{225}{228}$$

$$= 0.987$$

This is close to 1: there is a reasonably strong linear relation between x and y values — the line is a good fit.

4. $y(t) = -2.5t + 12.67$ implies $y(4) = 2.67$ and $y(4.5) = 1.42$
5. No sales means $y = 0$. Solve for t: $t = 5.068$ years = 61 months (rounded off)

Linear regression with Excel

Activity 4.15

Activity

Do a regression analysis

What will you do?

1 Repeat Activity 4.14 using Excel. Discuss as many regression statistics as you can.

2 Do a regression analysis for the following data from five companies and interpret the output.

Bonus of CEO (millions of CAD)	Profit of company (millions of CAD)
1.40	110.890
2.00	148.567
4.58	200.55
6.45	185.86
10.1	260.380

Solutions

1. We show part of the output table:

SUMMARY OUTPUT

Regression statistics	
Multiple R	0.993399
R square	0.986842
Adjusted R square	0.973684
Standard error	0.408248
Observations	3

ANOVA

	df	*SS*	*MS*
Regression	1	12.5	12.5
Residual	1	0.16666667	0.166667
Total	2	12.6666667	

	Co-efficients	*Standard error*	*t Stat*
Intercept	12.66667	0.62360956	20.31185
X Variable 1	-2.5	0.28867513	-8.66025

R and R square are close to 1, indicating good fit of linear relation.

Follow the Course Manual for further discussion.

2. Use the Bonus column as the Y range (dependent variable).

Regression statistics	
Multiple R	0.944596
R square	0.892261
Adjusted R square	0.856349
Standard error	1.342756
Observations	5

ANOVA

	df	*SS*	*MS*
Regression	1	44.795736	44.79574
Residual	3	5.4089836	1.802995
Total	4	50.20472	

	Co-efficients	*Standard error*	*t Stat*
Intercept	-5.86951	2.2436591	-2.61604
X Variable 1	0.059451	0.0119272	4.984495

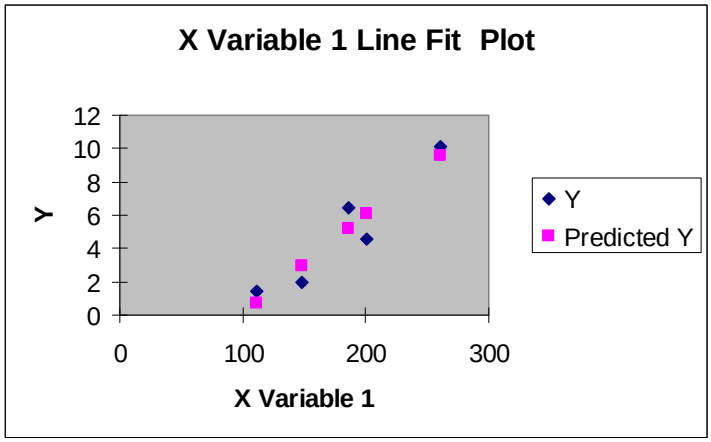
Note: Use the Bonus column as the Y range (dependent variable).

Use Profits as the X range (independent variable).

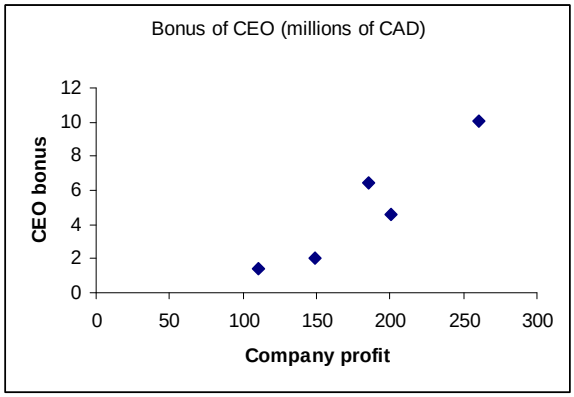
We note that the slope is 0.06 and the intercept at −5.9. The difference in scale between x and y causes the small slope. The value of R^2 is 0.89, which still indicates a reasonably good fit.

Graphs for linear regression (not in Course Manual)

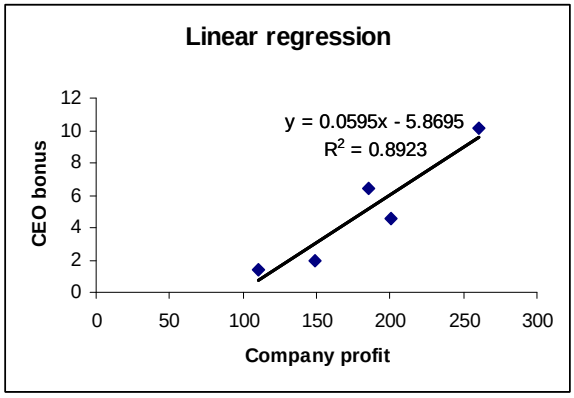
Instead of using Chart Wizard to draw the straight line from the equation, you can click the box "Line Fit Plots" in the Regression dialogue box. This produces the following, whose scale makes for a better graph.



Alternatively, use Chart Wizard (XY Scatter) to plot the separate data points.



Then right-click in the chart area. A new drop-down menu <Chart> will appear in the top toolbar next to <Tools>. Click on <Chart> and then click <Add Trendline>. A box will open. Choose "Linear" under the Type tab and then click the Options tab. Tick the two boxes "Display equation on Chart" and "Display R squared value on chart". Click <OK>. The graph will look like this:


Linear regression
y = 0.0595x - 5.8695
R2 = 0.8923
CEO bonus
Company profit
0
2
4
6
8
10
12
0
50
100
150
200
250
300

Assessment

Assessment — Module 4

1. Draw a binomial tree to represent the Markov price process $\{S_t : t = 0, 1, 2, 3\}$ for a share that starts with initial price $S_0 = 10$ and then either increases by 10% per time-step t or decreases by 10% per time-step t. The probability of an increase is 0.5 (50%).

   a) Is the tree recombining?

   b) Calculate $E[S_1]$ as well as $E[S_2]$ and compare the values with S_0.

   c) Repeat the calculations in b) above for the case where the probability of an increase is 0.7 (70%). What conclusions can you draw?

2. A fast-food company wants to study the arrival of customers at the counter. It has estimated the probabilities of certain numbers of customers arriving at the counter during any two-minute period. The data is:

Number of customers	0	1	2	3
Probability	0.1	0.45	0.25	0.2

   The company wants to simulate the number of customers arriving over 10 two-minute periods by generating strings of random numbers. The aim is to try to find out how many customers may arrive in a 20-minute period. This is important because it cannot cope if more than 18 people arrive during a typical 20-minute period.

   a) Generate a column of 10 random numbers using function RANDBETWEEN(0,99) in Excel. Discuss your method.

   b) Determine how many people arrived in a 20-minute period in this simulation.

   c) Repeat the process 20 times (a sample of 20 runs) and determine the average number of people that will arrive in a 20-minute period. Discuss your results and compare with those of fellow students.

3. A car manufacturer makes USD 1,000 profit on each model A car and USD 1,350 profit on each model B car. The assembly team can assemble a model A car in 30 minutes and a model B car in 45 minutes. A working day consists of eight hours.

   a) Write down an expression for the per-day profit function P. Use a spreadsheet package to represent this by a plane in 3-D space.

   b) Write down the daily time constraint. Represent the feasible region in a spreadsheet using Chart Wizard.

   c) Give a graphical representation of the feasible region showing the profit function contour line that gives maximum daily profit.

d) Find the optimal solution graphically.

e) Verify the solution with the corner-point method.

f) Use the Solver tool to find the optimal solution.

4. A profit function has the form $P(Q) = Q^3 - 2R^2 + 4QR - 3R$. The following constraints apply: $Q + R \geq 100$, $Q \leq 150$ and $R \leq 300$

   a) Find the values of Q and R that maximise the profit. Calculate the profit in this case.

   b) If there were no constraints, what would the maximum profit be? Use both analytical (calculus) methods and the Solver tool.

5. Your country's Internal Revenue Service (IRS) asks you to help develop a model to detect whether companies inflate the figures of their contributions to charity in order to gain greater deductions on their tax returns. You have the following data for eight companies whose tax returns have been audited and accepted as truthful.

   a) Draw a scatter graph for the data and decide whether a linear relationship is likely.

   b) Do a regression analysis and derive a linear equation for the relation between gross income (G) and charitable contribution (C).

   c) Interpret the value of R^2 and evaluate your model.

   d) Write a report to explain to the IRS how it could use your model to detect instances where companies claim unusually high contributions to charity.

Gross annual income (millions)	Contribution to charity
66.7	100,000
72.84	115,000
95.1	135,800
110.25	140,000
129.56	165,500
166.4	175,000
182.3	200,000
200.75	225,000

6. Consider the table of observed values y_t for six annual time steps t.

t	1	2	3	4	5	6
y_t	352.5	340	341.5	329.8	320	311.3

   a) Use the simple averages method to find a forecast for F_7.

   b) Use the 2-year moving averages method to find a forecast for F_7.

   c) Use the 4-year moving averages method to find a forecast for F_7.

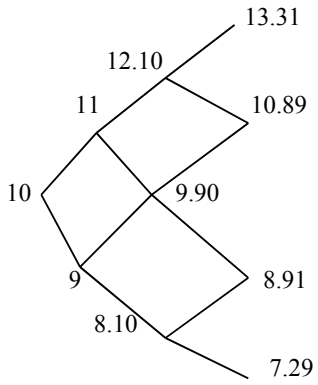
   d) Apply exponential smoothing with $\alpha = 0.1$ to find a forecast for F_7. Let $F_1 = 350$.

7. Consider the table of observed values y_t and forecast values $F1_t$ and $F2_t$. The forecasts F1 were obtained by Method 1 and the forecasts F2 were obtained by Method 2. Calculate some measures of error and interpret them. Which method is best?

t	1	2	3
y_t	30	34	38
$F1_t$	28	33	35
$F2_t$	28	35	40

Solutions

1.



t=0	t=1	t=2	t=3
S_0	S_1	S_2	S_3

   a) The tree is recombining.

   b) $E[S_1] = \frac{1}{2}(11 + 9)$
   $= 10$
   $= S_0$

$E[S_2] = ¼(12.10) + ½(9.90) + ¼(8.10)$
$= 10$
$= S_0$

We have a Markov process — in fact, a martingale process.

c) $E[S_1] = 0.7(11) + 0.3(9)$
$= 10.40$
$> S_0$

$E[S_2] = ¼(12.10) + ½(9.90) + ¼(8.10)$
$= 10.82$
$> S_0$

We have a Markov process, but not a martingale. The share price seems to be "drifting" upwards.

2. a) RANDBETWEEN(0,99) generates random numbers from 0 to 99. Each of the 100 possible numbers has a 1% probability of being generated.

The idea is that a number between 0 and 9 represents 0 people arriving in a 2-minute interval. Reason: there are 10 numbers from 0 to 9, which represent a 10% probability out of the 100 possible numbers from 0 to 99 that can be generated. Similarly, there are 45 numbers from 10 to 54, which represents a 45% probability out of the 100 possible numbers between 0 and 99. So numbers from 10 to 54 represent one person arriving in a 2-minute interval. Numbers from 55 to 79 represent two people arriving in a two-minute interval and numbers 80 to 99 represent three people arriving in a two-minute interval.

A column (or row) of 10 numbers gives 10 two-minute intervals – that is, 20 minutes. Translate the random numbers to numbers of customers and find the sum for this column to simulate how many customers may arrive in a 20-minute period.

b) Example of random numbers:

35	60	90	91	68	24	0	84	88	65

Translated numbers with sum:

										SUM
1	2	3	3	2	1	0	3	3	2	20

c) You can generate a block of size 20 × 10. Remember to translate each column (or row) of 10 to represent customers. Sum each column and find the average of the 20 numbers.

3. a) Let x denote number of model A cars assembled per day. Let y denote number of model B cars assembled per day. Profit $P(x, y) = 1{,}000\,x + 1{,}350y$

b) Constraints: $½\,x + ¾y \leq 8$, $x, y \geq 0$. We may treat this as a LP problem and allow fractions as answers, since we do not need to assemble a complete car in a day. Otherwise, if you want whole cars completed in a day, you can see this as an ILP problem.

The feasible region is a triangle with corner points (0, 0), (0, 10.667) and (16, 0).

c) and d)

Draw the triangle as described in b). The slope of the hypotenuse is −0.67; the slope of the profit contour lines is −0.74, which shows that contour lines have a steeper slope than the hypotenuse. This means the maximum profit contour line is the one touching point (16, 0).

Lecturers should review graphs for correctness.

e) Corner-point method: $P(0, 0) = 0$

$P(0, 10.667) = 1{,}350(10.667) = 14{,}400.45$

$P(16, 0) = 1{,}000(16) = 16{,}000$

The optimal solution for maximum profit is to assemble 16 model A cars per day and no model B cars.

Discussion: This is probably too simplistic and additional constraints may be necessary. For example, there may be demand for at least three model B cars per day. The added constraint $y \geq 3$ leads to a triangular feasible region with the corner points (0, 3), (0, 10.667) and (11.5, 3). The values of P are:
P (0, 3) = 4,050, P(0, 10.667) = 1,350(10.667) = 14,400.45 and P (11.5, 3) =15,550

The new optimal solution is 11.5 of model A and three of model B.

f) Solver used for the adapted problem with constraint $y \geq 3$:

P	15550
x	11.5
y	3
½x+¾y	8

Solver used with integer constraints on x, y:

P	15400
x	10
y	4
½x+¾y	8

1. a) Solver:

Profit	3419551
Q	150
R	149.25
q+r	299.25

b) No constraints: Solver cannot find a solution with starter values (0, 0). Remember the note in the Course Manual: *When you work with non-linear problems, it is best to not use starting values of zero (0) for all the relevant variables. Solver does not always like starting from 0.*

Try (−1, −1) as starting values.

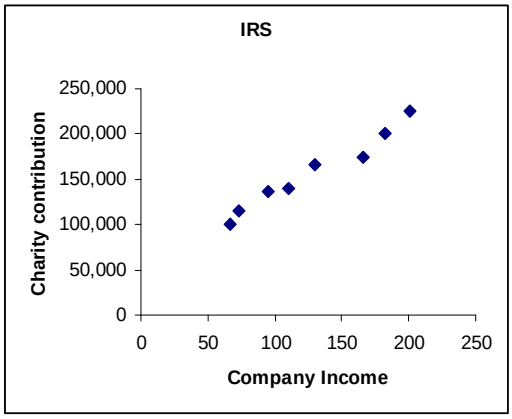
Profit	7.189605
q	-1.86852
r	-2.61852
q+r	-4.48703

Analytically: Find the derivatives of P for Q and to R. Set both equal to zero and solve. You should find:
$3Q^2 + 4R = 0$ and $-4R + 4Q - 3 = 0$

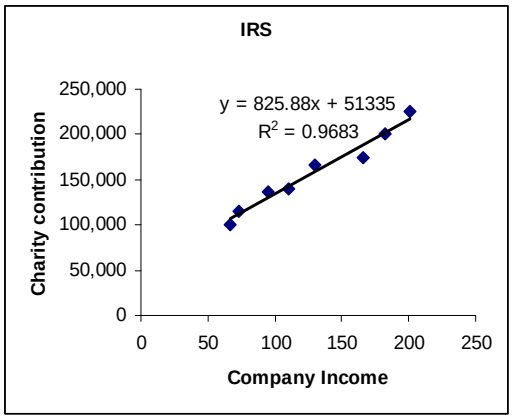
The first equation gives $4R = -3Q^2$. Substitute this into the second equation and solve the quadratic for Q. We obtain $Q = -1.867$ and
$Q = 0.54$. The second derivatives of P with respect to Q and R are $6Q$ and -4, respectively. Both should be negative for a maximum P. This is only possible for $Q = -1.867$.

$Q = 0.54$ leads to second derivatives of mixed sign and this means we have a saddle point at $Q = 0.54$.

5. a)



b) Click on the graph and use the Chart drop-down menu as explained previously. Otherwise, use <Tools> and <Data Analysis>.



c) The value of R^2 indicates a high linear relationship. For every 10 million increase in income, companies increase their contribution to charity by 8,258.80.

d) If a company claims it contributed 300,000 to charity, its gross income should be about 301 million. The IRS can now check what the company submitted as gross income.

6. a) $F_7 = 332.52$

b) $F_7 = \dfrac{\sum_{t=5}^{6} y_t}{2} = 315.65$

c) $F_7 = \dfrac{\sum_{t=3}^{6} y_t}{4} = 325.65$

d) $F_7 = \sum_{k=1}^{6} 0.1(0.9)^{k-1} y_{7-k} + (0.9)^6 (350)$ (with $\alpha = 0.1$)
$= 343.24$

The trend over the past four years is a decreasing one. One should therefore give the first values of y less weight. The four- or two-year moving averages should give better forecasts.

7. The data set is actually too small for a proper error analysis. However, calculations can get messy and we therefore use small data sets.

$\text{ME} = \dfrac{\sum_{t=1}^{3} e_t}{3}$ where $e_t = y_t - F_t$

$\text{MAD} = \dfrac{\sum_{t=1}^{3} |e_t|}{3}$

$\text{MSE} = \dfrac{\sum_{t=1}^{3} (e_t)^2}{3}$

Forecasting method 1:
Determine errors: $e_1 = 2$, $e_2 = 1$, $e_3 = 3$

ME = 2, MAD = 2, MSE = 4.67

Forecasting method 2:
Determine errors: $e_1 = 2$, $e_2 = -1$, $e_3 = -2$

ME = −0.33, MAD = 1.67, MSE = 3

Interpretation: For Method 1, ME > 0 shows that the forecasts are, on average, too low. For Method 2, ME < 0 shows that the forecasts are, on average, too high.

MAD gives the average distance of each forecast from the actual observation at that point.

MSE is useful in that it emphasises large errors and minimises small ones. The large error in Method 2 for forecast 3 has been amplified to give it a bigger MSE than Method 1. Method 2 is better overall.

Module 5

Decision-making, risk and challenges

Introduction

Outcomes

Upon completion of this module students will be able to:

- **evaluate** risk and risk aversion;
- **apply** utility theory to risk management;
- **solve** decision-making problems under uncertain conditions;
- **construct** and **use** decision trees for risky decision making;
- **analyse** and **evaluate** strategies using game theory;
- **appreciate** new challenges in management and economics (black swans and fat tails); and
- **critique** philosophies of economists and the new theory of behavioural economics.

Unit 17

Risk and decision-making

Outcomes

Upon completion of this unit students will be able to:

- **be aware of** the changes and challenges of the business in the 21st century;
- **understand** and **apply** the criteria of decision-making under uncertainty (Laplace, maximin, minimax and maximax); and
- **model** risky decision-making processes using decision trees.

Decision-making under certainty, risk and uncertainty

Activity 5.1

Activity
Apply maximax and minimax

What will you do?

Consider the situation of the events manager ordering food hampers. Apply the maximax and maximin decision criteria and give the chosen decision in each case. Discuss your results.

What do you think of the three criteria? Which suits your personality or business style?

Solutions

Pay-off table of profits

Profit pay-offs	Weather outcomes			
Strategies		Warm	Cool	Cold
	1. Buy large quantity	1200	600	−560
	2. Buy medium quantity	750	650	−120
	3. Buy small quantity	590	300	120

The maximax rule: Choose Strategy 1: Buy a large quantity for maximum profit of 1,200.

The maximin criterion: Minimum profit for strategies: −560, −120, 120. Maximum of these minima = 120. Choose Strategy 3: Buy a small quantity for profit of 120.

Activity 5.3

Activity

Final share decision

What will you do?

1. Calculate the volatilities of shares X, Y and Z. (Refer to Modules 2 and 4 if you have forgotten the definition of volatility.)
2. Repeat the calculations using Excel or Open Office.
3. What is your final decision about investing in one of the shares? Justify your answer.
4. Research the concept of diversification. Discuss the benefits of investing in more than one share.

Solutions

1. Volatility = standard deviation = $\sqrt{\text{var}}$

Mean = $\overline{X} = \sum x_i p(x_i)$ summed over all possible values x_i

Variance = $\text{var}(X) = \sigma^2(X) = \overline{X^2} - (\overline{X})^2$ where $\overline{X^2} = \sum x_i^2 p(x_i)$

In this case, the variable x_i of i=1, 2, 3 refers to the possible rates of return for a share.

vol(X) = 0.016, vol(Y) = 0.0181, vol(Z) = 0.0198

2. The answers should be the same.
3. Z has the highest volatility and is the riskiest. In general, higher returns imply higher risk. If you are very risk-averse, you may choose X instead.
4. It may be better to invest in a portfolio with a combination of the shares, provided they are negatively correlated. In such a case, you can decrease the overall volatility and still earn a good return.

Decision trees

Activity 5.4

Activity

Farm or mine?

What will you do?

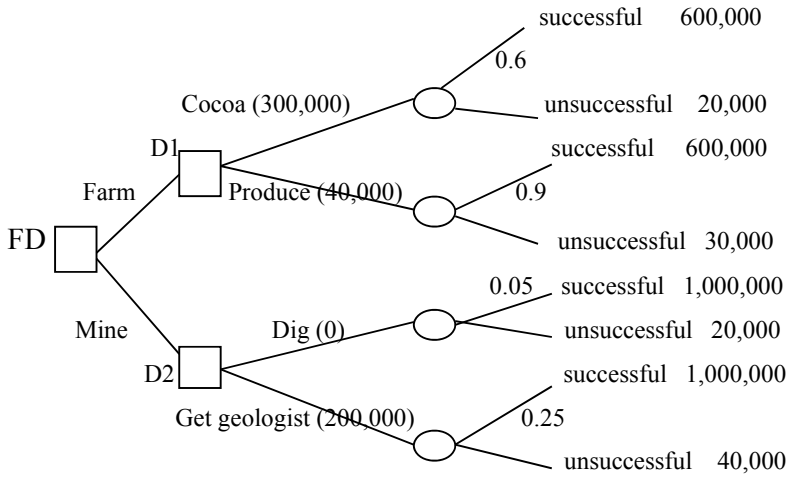
You have a piece of farmland where you think there may be diamonds. You have to decide whether to farm there or start mining. If you decide to farm, you can either plant cocoa for export or you can grow various produce for your own use and to sell locally. If you want to dig for diamonds, you can either get a geologist in to test for diamonds or just start digging.

The probability of a good outcome from deciding on diamonds and getting a geologist is 0.25. The value of this outcome is 1,000,000 and the value of a poor outcome is 40,000. The cost involved with a geologist is 200,000. The value of a positive outcome without a geologist is also 1,000,000 and the probability of a good outcome is 0.05. The value of a poor outcome is 20,000.

With cocoa, the costs are 300,000 and the probability of success is estimated at 0.6. The value of success here is 600,000. The value of no success here is 20,000. With produce, the costs are 40,000 and the probability of success is 0.9, with a final value of 600,000. The value of an unsuccessful outcome is 30,000.

Use a decision tree to analyse the situation. Explain your final decision and justify the course of action you will take.

Solution



Values of risky nodes (top to bottom): 368,000, 543,000, 59,000, 280,000

Values of decision nodes: D1 = 503,000, D2 = 80,000

Farming has far greater value than mining. Between cocoa and various produce, it is more expensive to get involved in cocoa farming and the chances of success are less.

Unit 18

Utility theory

Outcomes

Upon completion of this unit students will be able to:

- **express** values of outcomes in terms of utility functions;
- **rank** possible decisions with the help of utility theory;
- **calculate** certainty equivalents; and
- **determine** the Arrow-Pratt co-efficients for decision-makers.

Activity 5.9

Activity

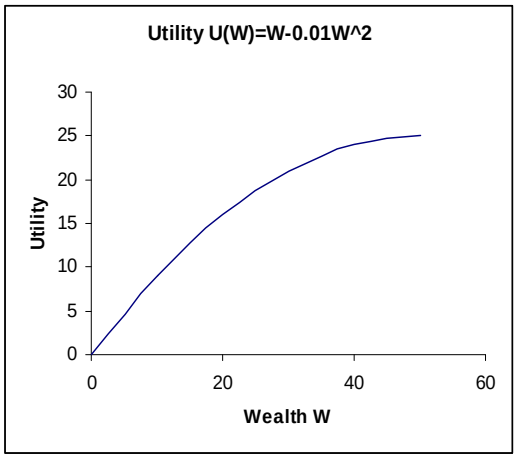
Work with the utility function

What will you do?

1. A manager is working with utility function $U(W) = W - 0.01W^2$. Two projects are being considered: Project A has outcomes of wealth 40 or 28, with probabilities of 0.45 and 0.55, respectively. Project B has outcomes of wealth 39 or 26, with probabilities of 0.6 and 0.4 respectively.
   a) Use Excel to generate a graph of function U (remembering that the values of W must be restricted to $W < 1/(0.02)$).
   b) Rank the projects and describe which project the manager should accept.
   c) Calculate the certainty equivalent to each project.
2. You are a broker and have analysed your client's risk aversion. Accordingly, you have constructed a utility function for her: $U(W) = -e^{-W}$
   a) Use Excel to generate a graph of function U.
   b) An investment opportunity offers wealth 2 with probability 0.45 and wealth 3 with probability 0.55. What is the client's certainty equivalent?
   c) Calculate the Arrow-Pratt risk co-efficient for the client. Discuss the implications.

Solutions

1. a)



The reason the graph is restricted to $W < 50$ is because U must remain concave with positive slope.

Slope = $\frac{dU}{dW} = 1 - 0.02W > 0$ if $W < 1/(0.02)$

b) Calculate expected utilities $E[U(W)]$

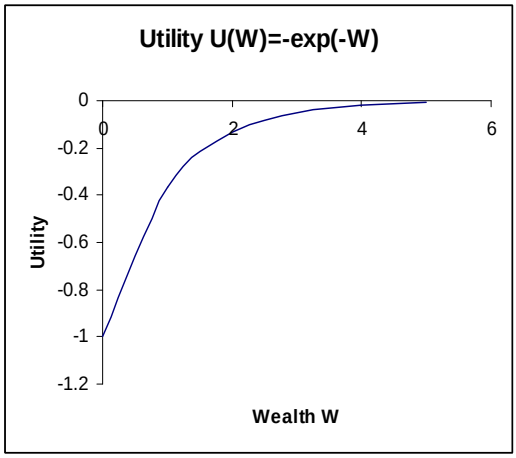
Project A: $E[U(W_A)] = 0.45\ U(40) + 0.55\ U(28)$
$= 10.8 + 11.09$
$= 21.89$

Project B: $E[U(W_B)] = 0.6\ U(39) + 0.4\ U(26)$
$= 14.27 + 7.7$
$= 21.97$

Project B ranks higher than A.

c) Certainty equivalent: $U(C) = E[U(W)]$ implies
$C - 0.01C^2 = 21.89$ for Project A and
$C - 0.01C^2 = 21.97$ for Project B
$C = 32.59$ for Project B and $C = 32.36$ for Project A.

2. a)



b) $U(C) = E[U(W)]$ implies $-e^{-C} = 0.45(-e^{-2}) + 0.55(-e^{-3})$

$= -0.0883$

$e^{-C} = 0.0883$

$\ln e^{-C} = \ln 0.0883$

$-C = -2.43$

$C = 2.43$

c) A-P $= -\dfrac{U''(W)}{U'(W)} = 1$ ($U'' = (-e^{-W}$ and $U' = e^{-W}$)

As wealth level W increases or decreases, A-P remains constant. The client will therefore remain at the same level of risk-averseness.

Unit 19

Game theory

Outcomes

Upon completion of this unit students will be able to:

- **set up** pay-off tables for strategies in game theory; and
- **analyse** situations with dominating strategies**.**

Activity 5.11

Activity

Make sure you win

What will you do?

Use the data in the example to answer these questions.

1. Construct the pay-off table for Player Odd.
2. Check that it is a zero-sum game.
3. Is there a specific strategy that a player can follow to definitely win (maximise their utility)? Alternatively, can they co-operate so that both are happy?

Solutions

1.

Pay-off for Odd	Player Odd		
Player Even	Strategy	S1	S2
	S1	−1	+1
	S2	+1	−1

2. The sum of corresponding entries in tables for the two players is zero.
3. There is no strategy and no possible co-operation. One player has to win and the other player will lose, unless both decide not to play at all.

Case study:
Retail strategies

Consider the problem of the two managers trying to win customers (the dominating strategies example). Use the same pay-off table or your own data.

1. Manager 1 chooses to ignore the fact that there is a second competing manager who can strategise. She sees her pay-off table purely in terms of uncertain decision making, as in Unit 17. Use the four criteria and find the decision she will make in each case. How do these decisions compare with the case where she sees the situation in terms of game theory?
2. Suppose Manager 2 chooses to ignore the fact that there is a second competing manager who can strategise. He, too, sees his own pay-off table purely in terms of uncertain decision making, as in Unit 17. Use the four criteria and find the decision he will make in each case. How do these decisions compare with the case where he sees the situation in terms of game theory?
3. Compare and discuss the methods of decision making in Unit 17 with the game theory method.
4. *Optional project (not teacher-assessed):* Devise a spreadsheet that calculates the maximin and minimax criteria. Hint: Use Excel functions such as MIN, MAX and IF.

Solution

1.

Pay-off for Manager 1	Manager 2			
Manager 1	Strategy	S1	S2	S3
	S1	100	200	−50
	S2	50	150	−200
	S3	90	90	20

Laplace: Calculate expected pay-offs for each strategy.

Strategy 1: ⅓(100) + ⅓(200) + ⅓(−50) = 83.33 (rounded off)

Strategy 2: ⅓(50) + ⅓(150) + ⅓(−200) = 0

Strategy 3: ⅓(90) + ⅓(90) + ⅓(20) = 66.67 (rounded)
Decision: Follow Strategy 1.

Minimax: Maximum losses are 50, 200 and −20 for strategies 1, 2 and 3 respectively. The minimum is −20.
Decision: Follow Strategy 3.

Maximax: Maximum profits are 200, 150 and 90 for strategies 1, 2 and 3 respectively. The maximum is 200.
Decision: Follow Strategy 1.

Maximin: Minimum profits are −50, −200 and 20 for strategies 1, 2 and 3 respectively. The maximum is 20.
Decision: Follow Strategy 3.

Game theory, minimax and maximin (here) give the same result (Strategy 3).

2. Do the same for Manager 2 with the pay-off table:

Pay-off for Manager 2	Manager 2			
Manager 1	Strategy	S1	S2	S3
	S1	-100	-200	50
	S2	-50	-150	200
	S3	-90	-90	-20

Examples:

Minimax: Maximum losses are 100, 200 and −20 for strategies 1, 2 and 3 respectively. The minimum is −20.
Decision: Follow Strategy 3.

Maximax: Maximum profits are: −50, −90 and 200 for strategies 1, 2 and 3 respectively. The maximum is 200.
Decision: Follow Strategy 3.

Manager 2 always follows Strategy 3.

3. Zero-sum game theory and the minimax criterion should give the same result.
4. Optional question. Lecturers to assess student responses.

Unit 20

New challenges: Risk and human behaviour

Outcomes

Upon completion of this unit students will be able to:

- **understand** the role of rare events of huge impact;
- **critique** some thoughts and philosophies that impact on quantitative techniques;
- **apply** value at risk (VaR) calculations;
- **discuss** the limitations of VaR; and
- **appreciate** the importance of human behaviour in the modelling of economic processes.

Activity 5.13

Activity

What is the maximum loss?

What will you do?

A company plans to invest its profits of 55 million rupees in a financial product for one year. The possible profits are very large, but there may be losses. The chief executive is happy to take the risk as long as the expected loss is not greater than 1 million rupees.

Consider the scenario where a manager identifies three possible events that may cause losses.

Event 1: Loss of 0.5 million rupees with probability 0.551

Event 2: Loss of 1.5 million rupees with probability 0.448

Event 3: Loss of X million rupees with probability 0.001

What is the maximum amount X that the company could lose if the rare Event 3 is realised? (Hint: Set the expected loss equal to 1 million and solve for X.)

Solution

If we leave Event 3 out of the picture, the expected loss E[L] is 0.948 million rupees. This is acceptable to the chief executive. Managers may be tempted to forget about Event 3, since the probability is so small. Let's assume an expected loss of 1 million and include Event 3. The equation is:

$$E[L] = 0.5 \times 0.551 + 1.5 \times 0.448 + X \times 0.001 = 1$$

Then $X = 52.5$ million rupees. In other words, merely setting a limit of 1 million on expected losses can lead to implicit acceptance of a possible loss of 52.5 million. Event 3 would represent a typical black swan.

Swans and value at risk

Activity 5.14

Activity

Calculate VaR

What will you do?

A company plans to invest its profits of 100 million rupees in shares in an oil company. The annual volatility of the share price is 58 per cent.

1. Calculate the 10-day 99 per cent VaR of the investment.
2. Calculate the 10-day 95 per cent VaR of the investment.

Discuss your answers.

Solutions

1. We first calculate the 1-day 99% VaR. The daily volatility is: $\frac{0.58}{\sqrt{252}} = 0.0365$. This means that over 1 day the standard deviation of our investment is 3.65% of 100 million or:

   $0.0365 \times 100{,}000{,}000 = 3{,}650{,}000$ rupees

   The 1-day 99% VaR for our position is therefore:

   $2.33 \times 3{,}650{,}000 = 8{,}504{,}500$

   The 10-day 99% VaR for our position is therefore:

   $8{,}504{,}500 \times \sqrt{10} = 26{,}893{,}590$ rupees (rounded off)

   We can be 99% sure that we will not lose more than 26,893,590 rupees over a period of 10 days, or there is a 1% chance that we will lose that amount or more over a period of 10 days.

2. The 1-day 95% VaR for our position is therefore:

   $1.96 \times 3{,}650{,}000 = 7{,}154{,}000$

   The 10-day 99% VaR for our position is therefore:

   $7{,}154{,}000 \times \sqrt{10} = 22{,}622{,}934$ rupees (rounded off)

   We can be 95% sure that we will not lose more than 22,622,934 rupees over a period of 10 days or there is a 5% chance that we will lose that amount or more over a period of 10 days.

Economic development: Choice and behaviour

Activity 5.15

Activity

Make a rational decision

What will you do?

Look at the "Rank the options" example and explain why no majority decision can be made based on the rankings.

Now assume Manager 1's ranking carries weight 1, Manager 2's ranking carries weight 1.5 and Manager 3's ranking carries weight 3.

Can you make a rational decision?

Solution

Example 1: Manager 1's ranking is: $x > y > z$

Manager 2's ranking is: $y > z > x$

Manager 3's ranking is: $z > x > y$

Any pairing of managers leads to a conflicting ranking. Managers 1 and 2 have a conflict on the ranking of x and z. Managers 2 and 3 have a conflict on ranking y and z. Managers 1 and 3 have a conflict on ranking y and z. For this reason, a majority is impossible.

Manager 3 has a vote larger than the combined votes of managers 1 and 2. Her ranking will be accepted.

Case study:
Group research project

This case study involves rational behaviour in people's decision making.

1. Analyse these situations and show that, rationally speaking, people who choose Option 1 in Situation X should also choose Option 1 in Situation Y. Likewise, people who choose Option 2 in Situation X should also choose Option 2 in Situation Y.

   Situation X

   You are given 10,000 units of money and must choose between two courses of action:

   - Option 1: Throw a dice. If it shows an even number, you win 5,000 units. If it shows an odd number, you win nothing.
   - Option 2: Receive 2,000 units with certainty.

   Situation Y

   You are given 15,000 units of money and must choose between two courses of action:

   - Option 1: Flip a coin. If it shows heads, you lose 5,000 units. If it shows tails, you lose nothing.
   - Option 2: Lose 3,000 units with certainty.

2. Send out a questionnaire to a group of people, asking them to make a choice between the options in Situation X. Analyse the results. Then send out another questionnaire to the same group of people, asking them to make a choice between the options in Situation Y. Write up your results as a report.

Discussion

People who choose Option 1 in situation X: Expected pay-off = 12,500

People who choose Option 1 in situation Y: Expected pay-off = 12,500

People who choose Option 2 in Situation X: Expected pay-off = 12,000

People who choose Option 2 in Situation Y: Expected pay-off = 12,000

Assessment

Assessment — Module 5

1. As manager for a retail store, you have to order new stock. There are three ranges to choose from: luxury, middle of the range and inexpensive. The strategic decision on which range to buy depends on the economic outlook for the season, but the economic forecasts are uncertain. Historical data shows that the expected profits are:

Profit pay-offs	Economic outlook			
Strategies		Good	Average	Bad
	1. Buy luxury range	100,000	40,000	−80,000
	2. Buy middle of range	70,750	60,000	−30,200
	3. Buy inexpensive range	−20,200	50,500	50,000

Figure 8 Pay-off table of profits

   a) Use the Laplace criterion to make a decision.
   b) Use the Wald criterion to make a decision.
   c) Use the maximax criterion to make a decision.
   d) Use the maximin criterion to make a decision.
   e) Compare and discuss the results.

2. Assume that further research shows probabilities can be assigned to the possible economic outcomes in Question 1. The probabilities are given in Figure 14.

   Use expected profit values to decide on the strategy you will follow to order stock under the given risks.

Good economy	Average economy	Bad economy
0.2	0.34	0.46

Figure 9 Table of probabilities

3. You have a piece of land where you think there may be oil. You have to decide whether to have a seismic survey done or not. Seismic surveys are reasonable accurate, but not always. The cost of a survey is 100 monetary units (which can be in thousands of pounds or

dollars).

Regardless of the survey, your next decision is to drill for oil or sell the land. The cost of drilling is 200 units and the income from selling is 100. (These two values will have different signs in the tree.)

If you decide to drill after a survey, the outcomes are either a dry hole with probability 0.5 or oil. If you decide to drill without a survey, the outcomes are a dry hole with probability 0.7 or oil. The final pay-off for oil is 1,000 and the final pay-off for a dry hole is −300.

If you decide to sell, there are no further risky outcomes and the final pay-off in all cases is 100.

Use a decision tree to analyse the situation. Discuss the outcomes of the various options. Write up your final decision and justify the course of action you will take.

4. Two managers are in a competitive zero-sum situation. The employees from their divisions can accept new positions in the opposite division. That is, employees from Manager 1's division can accept a job in Manager 2's division and vice versa. Each manager wants to attract as many people from the other division and lose as few as they can. Each manager has three strategies they can follow. The pay-off matrix for a particular manager is constructed as follows: positive numbers indicate the number of employees gained from the other division and negative numbers indicate the number of employees lost to the competing manager's division.

   The strategies and the pay-offs are shown in the table below. Pay-offs are from the perspective of Manager 1. In other words, if Manager 1 follows his strategy 1 and Manager 2 follows her strategy 1, then two employees will leave Manager 1's division and join Manager 2's division. But if Manager 1 follows his strategy 1 and Manager 2 follows her strategy 3, then four people will join Manager 1 and leave Manager 2.

   The pay-off table for Manager 1 is:

Pay-off for Manager 1	Manager 2			
Manager 1	Strategy	S1	S2	S3
	S1	−2	−2	4
	S2	2	−1	3
	S3	−1	−2	2

Figure 10

   a) Are there any dominating strategies for Manager 1?

   b) Draw the pay-off matrix for Manager 2.

   c) Are there any dominating strategies for Manager 2?

d) Use the process of dominating strategies to identify the strategy each manager should follow.

5. Assume you have utility function $U(W) = 0.5\sqrt{W}$ where W indicates wealth level. You can invest in a risky project for one year.

   The risky project has three outcomes in wealth: 1,600 pounds with probability 0.5, 1,800 pounds with probability 0.3 and 2,200 pounds with probability 0.2.

   a) What is the expected utility of wealth for the project?

   b) Calculate the certainty equivalent for the project.

   c) The project needs an initial investment of 1,500 pounds. As an alternative to investing in the risky project, you can also invest the money in the bank for a year at 16.5 per cent p.a. compound interest.

   Will you accept the project or not?

6. A financial manager is working with utility function $U(W) = W - 0.01W^2$. Find an expression for the Arrow-Pratt risk-aversion co-efficient and discuss the implications. How does the co-efficient change with changing values of W?

7. A company plans to invest profits, of 50 million Malaysian ringgit, in shares in a construction company. The annual volatility of the share price is 36 per cent.

   a) Calculate the daily volatility of the share price.

   b) Calculate the 10-day 99 per cent VaR of the investment.

   c) If the company cannot accept losses on an investment of more than 5 million ringgit, should it invest in the shares? Discuss your answer.

Solutions

1. a) **Laplace:** Calculate expected pay-offs for each strategy. Units in thousands are used:

   Strategy 1: ⅓(100) + ⅓(40) + ⅓(−80) = 20

   Strategy 2: ⅓(70.75) + ⅓(60) + ⅓(−30.2) = 33.52

   Strategy 3: ⅓(−20.2) + ⅓(50.5) + ⅓(50) = 26.77
   Decision: Follow Strategy 2.

   b) **Wald (minimax)**: Maximum losses are 80, 30.2 and 20.2 for strategies 1, 2 and 3, respectively. The minimum is 20.2.
   Decision: Follow Strategy 3.

   c) **Maximax:** Maximum profits are 100, 70.75 and 50.5 for strategies 1, 2 and 3, respectively. The maximum is 100.
   Decision: Follow Strategy 1.

d) **Maximin**: Minimum profits are −80, −30.2 and −20.2 for strategies 1, 2 and 3, respectively. The maximum is −20.2. Decision: Follow Strategy 3.

Discussion: Different criteria give different results. Lecturers should engage students in discussion.

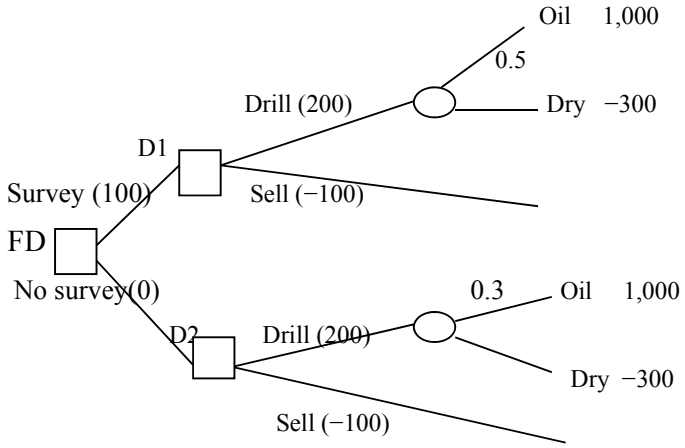
2. Strategy 1: E[profit] = −3.2

   Strategy 2: E[profit] = 20.66

   Strategy 3: E[profit] = 34.11

   Follow Strategy 3.

3.



Values of risky nodes (top to bottom): 350, 90

Value of decision nodes: D1 = max[350−200, −(−100)] = 150
D2 = max[90−200, 100)] = 100

FD = max[150−100, 100 – 0] = 100

We should decide to do a survey and drill rather than no survey and drill. But if we cannot afford a survey, then we should sell.

4. a) Strategy 2 dominates Strategy 3. Remove Strategy 3 for Manager 1.

b)

Pay-off for Manager 2	Manager 2			
Manager 1	Strategy	S1	S2	S3
	S1	+2	+2	−4
	S2	−2	+1	−3

c) Strategy 2 also dominates Strategy 3 for Manager 2.

d) Remove Strategy 3 for Manager 2. Manager 1 will prefer Strategy 2. Manager 2 will also prefer his Strategy 2. Both follow their Strategy 2. Manager 1 loses one employee and Manager 2 gains that employee.

5. a) $E[U(W)] = 0.5(0.5)\sqrt{1600} + 0.3(0.5)\sqrt{1800} + 0.2(0.5)\sqrt{2200}$

$= 21.05$

b) $0.5\sqrt{C} = 21.05$ implies $C = 1{,}772.41$

c) The bank offers $1500(1.165) = 1{,}747.50$ with certainty. This is less than the certainty equivalent, so you should accept the project.

6. A-P $= -\dfrac{U''(W)}{U'(W)}$ with $U'(W) = 1 - 0.02W$ and $U''(W) = -0.02$

$$\text{A-P} = -\frac{-0.02}{1-0.02W} = \frac{0.02}{1-0.02W}$$

As W increases, the co-efficient increases, slowly at first but very quickly as W tends to 50. Risk aversion thus increases with increasing wealth.

7. a) 1-day volatility $= \dfrac{0.36}{\sqrt{252}} = 0.0227 = 2.27\%$

b) Over 1 day, the standard deviation of our investment is 2.27% of 50 million, or $0.0227 \times 50{,}000{,}000 = 1{,}135{,}000$ ringgit

The 1-day 99% VaR for our position is therefore
$2.33 \times 1{,}135{,}000 = 2{,}644{,}550$

The 10-day 99% VaR for our position is therefore
$2{,}644{,}550 \times \sqrt{10} = 8{,}362{,}801$ ringgit (rounded off)

c) We can be 99% sure we will not lose more than 8,362,801 ringgit over a period of 10 days, or there is a 1% chance that we will lose that amount or more over a period of 10 days. The VaR is more than 5 million. The company will have to decide whether it wants to take the risk.

Final examinaton: (model questions)

The examination paper below should be seen as an example of an exam. The questions can be adapted to suit particular circumstances. There are many examples of other possible questions in the various activities and summative assessments in each module.

The duration of the exam should also be taken into account, and the paper shortened or lengthened accordingly. It is a good idea for the lecturer to work through the questions to determine how many minutes are needed for each question.

The model exam below is based on the assumption that it is a written closed-book exam, with the use of a pocket calculator permitted.

It is assumed that access to Excel is not available during the exam and for this reason there are no questions based on the use of Excel in this paper. Marks for activities involving Excel can be collected during the run of the course and added to the exam mark if necessary. If Excel is made available for the exam, the lecturer can adapt the paper by including questions on Excel. Once again, there are many examples of questions available in the activities.

Final assessment — Example of an examination

1. Let $C(x)$ be the cost of producing x items. The marginal cost MC $(= \frac{dC}{dx})$ is given by $MC(x) = x - 1$ and the initial start-up cost before production starts is $C(0) = 1$.

   a) Determine the general expression for cost $C(x)$. [3]

   b) Calculate the cost to produce the first 10 items. [2]

   c) The income function $I(x)$ is known to be linear. There are two data points available: (10, 40) and (0, 20). Find the expression for $I(x)$. [3]

   d) Draw graphs of both $C(x)$ and $I(x)$ on the same set of axes. The divisions along the x-axis should be scaled in units of 2. Determine the approximate break-even point graphically. [4]

   e) Determine the break-even point analytically. [4]

   [16]

2. The following frequency table gives the sizes of fishing vessels (boats) and the number of vessels in each class on an island for the years 1992 and 2002.

Length in metres	*1992*	*2002*
3.01 to 10 m	76	27
10.01 to 17 m	237	123
17.01 to 24 m	419	451
24.01 to 31 m	200	245
31.01 to 38 m	115	60
Source: Annual Abstract of Statistics (UK)		

   a) What conclusions can you draw about the state of fishing on the island? [1]

   b) Expand the table by adding a column for midpoints of intervals and a column for cumulative frequencies for the two years. Midpoints should be rounded to 1 decimal. [4]

   c) Find the means for the two years. Discuss your answers. [5]

   d) Draw a histogram for the distributions. [4]

   [14]

3. Random variable X is binomially distributed and describes the number of successful outcomes in five trials. The probability of a success is 0.7.

   a) Calculate the mean and variance of X. [4]

   b) Determine the probability of, at most, two successes. [4]

   [8]

4. Random variable X is normally distributed on a population. Consider a sample of 200 from the population. The sample has mean 0 and variance $s^2 = 16$.

   a) Calculate the 95 per cent confidence interval for X. Discuss the meaning of this interval. [5]

   b) What is the probability that $X > 2$? [3]

   [8]

5. Consider the table of observed values y_t for six annual time steps t. The row of forecast values F_t was obtained by the 2-year moving averages method.

t	1	2	3	4	5	6
y_t	352.5	340	341.5	329.8	320	321.3
F_t	350.5	350	346.3	340.8	335.7	324.9

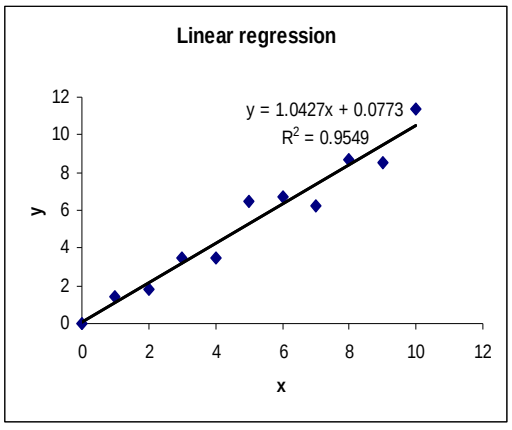
   a) Use the simple averages method to find forecasts for F_1 to F_6. Let $F_1 = 348$. [5]

   b) Use the four-year moving averages method to find a forecast for F_7. [2]

   c) Calculate the mean squared error for each of the simple averages and two-year moving averages method. Compare and interpret your answers. [5]

   [12]

6. Consider the following output for a linear regression problem. Data points are indicated by diamond shapes.



   a) Explain the values in the chart. [4]

   b) Estimate a value for y if $x = 20$. [1]

   [5]

7. A company manufactures garden forks and spades. It costs 10 dollars to make a spade and 8 dollars to make a fork. There is an additional

daily cost of 100 dollars per day for labour. Denote the daily manufactured number of forks by x and the number of spades by y. It takes five minutes to make a spade and five minutes to make a fork. The company has eight hours per day for manufacturing. It wants to minimise costs, but must manufacture at least 30 items per day to supply customers. It must also manufacture fewer than 20 spades per day.

a) Write down an expression for the daily cost function C. [2]

b) Write down the daily constraints. [2]

c) Give a graphical representation of the feasible region. [2]

d) Find the optimal solution using the corner-point method. Discuss. [4]

[10]

8. As manager for a store, you have to embark on new strategies to increase your profits. There are three possible strategies and the profits for each depend on two possible scenarios: high inflation, or low inflation. Historical data shows that the expected profits are:

Pay-off table of profits

Profit pay-offs	Inflation outlook		
Strategies		High	Low
	1.	1,000	2,000
	2.	775	2,800
	3.	−200	3,500

a) Assume complete uncertainty for the inflation outlook. Use the Laplace criterion to make a decision. [3]

b) Use the Wald criterion to make a decision. Compare with the answer in a). [3]

c) Now assume probabilities can be assigned to the inflation outcomes: the probability of high inflation is 0.65 as estimated from data. Which strategy will you follow in this case if you use expected profits as benchmark? [3]

[9]

9. Assume you have utility function $U(W) = \ln W$ where W indicates wealth level. You have 1,000 dollars to invest for two years, either in the bank (risk-free) or in a risky project.

The risky project has three outcomes in wealth: 1,800 dollars with probability 0.1, 1,400 dollars with probability 0.6 and 1,200 dollars with probability 0.3.

a) What is the expected utility of wealth for the risky project? [3]

b) Calculate the certainty equivalent for the risky project. [4]

c) The risk-free investment is in the bank at 10% p.a. interest, compounded semi-annually. Will you accept the risky project or not? [4]

d) Find an expression for the Arrow-Pratt risk-aversion co-efficient and discuss the implications. [4]

[15]

10. A company has invested 12 million rupees in shares. The annual volatility of the share price is 48%.

a) Calculate the daily volatility of the share price. [2]

b) Calculate the 10-day 99% VaR of the investment. What does the figure mean? [4]

[6]

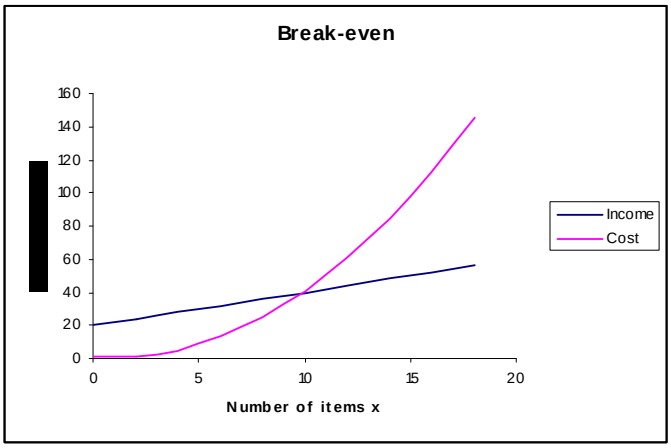
Solutions

1. a) $C(x) = \int(x-1)dx = \frac{1}{2}(x)^2 - x + k$. Since $C(0) = 1$ we have $k = 1$ and $C(x) = \frac{1}{2}(x)^2 - x + 1$

b) $C(10) = \frac{1}{2}(10)^2 - 10 + 1 = 41$ monetary units

c) Income function I must be of the form: $I(x) = mx + c$. Given data point (0, 20) means $I(0) = 20$ and thus we have $c = 20$. Substitute (10, 40) into equation $I(x) = mx + 20$ gives: $40 = 10m + 20$ and so $m = 2$. Therefore $I(x) = 2x + 20$

d)



e) Break-even means $I(x) = C(x)$. This leads to the quadratic equation: $x^2 - 6x - 38 = 0$

The two roots are $x = -3.86$ and $x = 9.86$.
Break-even occurs when about 10 items are manufactured.

2. a) The fishing industry has decreased slightly over 10 years in terms of numbers of boats. However, there are now more boats of the mid-size range. Small fishermen may be going out of business.

b)

Midpoints	1992 Cum. *F*	2002 Cum. *F*
6.5	76	27
13.5	313	150
20.5	732	601
27.5	932	846
34.5	1047	906

c) 1992: Mean = $\frac{1}{N}\sum_{i=1}^{N} x_i f(x_i)$

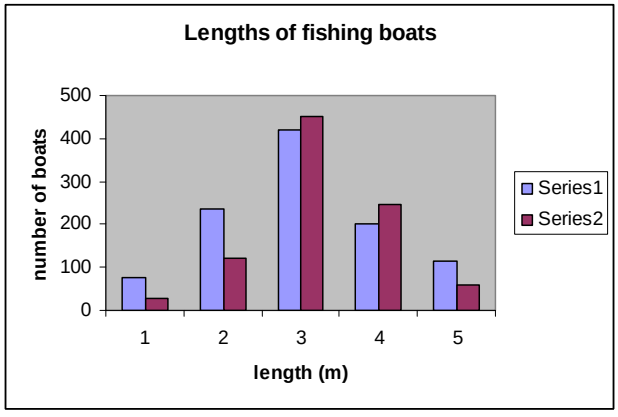
= $\frac{1}{1047}$ [6.5(76) + 13.5(237) + 20.5(419) + 27.5(200) + 34.5(115)] = 20.8 metres (rounded)

2002: Mean = $\frac{1}{N}\sum_{i=1}^{N} x_i f(x_i)$

= $\frac{1}{906}$ [6.5(27) + 13.5(123) + 20.5(451) + 27.5(245) + 34.5(60)]
= 22 metres (rounded)

The average length of boats has increased.

d)



3. a) Mean = $\overline{X}$ = 5 × 0.7 = 0.35
var(X) = $\sigma^2(X)$ = 5× 0.7 × 0.3 = 1.05

a) The probability of, at most, two successes is:
$P(X = 0 \text{ or } X = 1 \text{ or } X = 2) = p(0) + p(1) + p(2)$

Now $p(0) = \binom{5}{0}(0.7)^0(0.3)^5 = 1\times1\times(0.3)^5 = 0.0024$

$p(1) = \binom{5}{1}(0.7)^1(0.3)^4 = 5\times(0.7)\times(0.3)^4 = 0.0284$

$p(2) = \binom{5}{2}(0.7)^2(0.3)^3 = 10\times(0.7)^2\times(0.3)^3 = 0.1323$

Therefore the probability is 16.31%.

4. a) The 95 per cent confidence interval is:
$(0 - 1.96(4)/\sqrt{200},\ 0 + 1.96\ (4)/\sqrt{200}) = (-0.55, 0.55)$
This means the population mean will lie in this interval around 0 with 95% certainty.

   b) $P(X > 2) = P(X > \sigma) = \frac{1}{2}(1 - P(-\sigma < X < \sigma)) = \frac{1}{2}(1 - 0.68) = 0.16 = 16\%$

5. a)

t	1	2	3	4	5	6
y_t	352.5	340	341.5	329.8	320	321.3
F_t	350.5	352.5	346.3	344.7	340.9	336.8

   b) $F_7 = 328.2$

   c) Two-year moving averages MSE = 84.6
Simple averages MSE = 180.4
The simple averages method is much worse than the moving averages method. The reason is that there is a declining trend and it is therefore better to use only the last few values to capture the trend.

6. a) The R^2 value of 0.9549 shows quite a strong linear relationship between the x- and y-values. The graph also shows the data points lying close to the straight line. The slope of the line is approximately 1, showing an increase of about 1 unit in y for every 1-unit change in x. The y-intercept is close to 0. This, together with the slope of 1.0427, implies that the x- and y-values are almost equal.

   b) $y(20) = 1.0427(20) + 0.0773 = 20.93$

7. a) $C(x, y) = 8x + 6y + 100$

   b) $5x + 5y \le 480$, $x + y \ge 30$, $y \le 20$, $x, y \ge 0$. We can also add integer constraints.

   c) The feasible region is the trapezium-shaped region between the parallel lines $y = 96 - x$ and $y = 30 - x$ and the parallel lines $y = 20$ and $y = 0$.

   Lecturers should review graphs for correctness.

d) The corner points are: (10, 20), (30, 0), (76, 20), (0, 96)
C has a minimum at (10, 20). This means that 10 forks and 20 spades should be manufactured to minimise costs. It must be noted that this solution is not efficient in the sense that workers will be busy for only 150 minutes per day

8. a) Strategy 1: E[profit] = ½ [1,000] + ½ [2,000] = 1,500
Strategy 2: E[profit] = ½ [775] + ½ [2,800] = 1,787.50
Strategy 3: E[profit] = ½ [−200] + ½ [3,500] =1,650
Therefore, follow Strategy 2.

b) Rewrite table in terms of losses. Find maximum losses for each strategy and then the minimum. Therefore, follow Strategy 1 (the most conservative strategy).

c) Strategy 1: E[profit] = 0.65[1,000] + 0.35[2,000] = 1,350
Strategy 2: E[profit] = 0.65[775] + 0.35[2,800] = 1,483.75
Strategy 3: E[profit] = 0.65[−200] + 0.35[3,500] =1,095
Follow Strategy 2.

9. a) E[$U(W)$] = 0.1[ln(1800)] + 0.6[ln(1400)] + 0.3[ln(1200)] = 7.223

b) $U(C)$ = 7.223
ln C = 7.223
C = 1,370.59

c) Risk-free: $1000(1 + 0.05)^4$ = 1,215.51
This is less than the certainty equivalent. Invest instead in the risky project.

d) A-P co-efficient: $U'(W) = \frac{1}{W}$ and $U''(W) = \frac{-1}{W^2}$

A-P = $\frac{1}{W}$. As wealth increases, risk aversion decreases. The investor will thus be inclined to make more risky investments as her wealth grows.

10. a) 1-day volatility = $\frac{0.48}{\sqrt{252}}$ = 0.0302 = 3.02%

b) Over 1 day the standard deviation of our investment is 3.02% of 12 million, or 0.0302 × 12,000,000 = 362,400 rupees. The 1-day 99% VaR for the position is therefore:

2.33 × 362,400 = 844,392
The 10-day 99% VaR for our position is therefore:
844,392 × √10 = 2,670,202 rupees (rounded off)

We are therefore 99% sure that we will not lose more than 2,670,202 rupees over a period of 10 days, or there is a 1% chance that we will lose that amount or more over a period of 10 days. The company will have to decide whether it wants to take the risk.